Strong induction

Ex. Suppose \( a_0 = 0 \), \( a_1 = 3 \), and \( a_{n+1} = a_n + 2a_{n-1} \). Find \( a_{100} \).

Solution. Let’s start with inductive reasoning to guess what \( a_n \) should be. [I am too lazy to compute \( a_{100} \) by hand.]

\[
0, \ 3, \ 3, \ 9, \ 15, \ 33, \ 63, \ 129, \ ...
\]

Any guess? If not, let’s do one more: 255.

Are these numbers similar to the ones in 2048 game!? 2^0 - 1, 2^1 - 1, 2^2 - 1, 2^3 - 1, 2^4 - 1, 2^5 - 1, 2^6 - 1.

Conjecture \( a_n = \begin{cases} 2^{n-1} & \text{if } n \text{ is even} \\ 2^{n+1} & \text{if } n \text{ is odd} \end{cases} \)

i.e. \( a_n = 2^n - (-1)^n \).

If this conjecture has an affirmative answer, we get \( a_{100} = 2^{100} - 1 \).

How can we prove this conjecture?
Since \( a_n \) is defined recursively, it is a good idea to try induction.

\[
\begin{align*}
a_{k+1} &= a_k + 2a_{k-1} \quad \text{unless} \quad k = 0 \text{ or } 1 \\
&= (2^k - (-1)^k) + 2 \left(2^{k-1} - (-1)^{k-1}\right) \\
&= 2^k - (-1)^k + 2^k - 2(-1)^{k-1} \\
&= (2^k + 2^k) - \left[-(-1)^{k+1} + 2(-1)^{k-1}\right] \\
&= (2)(2^k) - (-1)^{k+1} \\
&= 2^{k+1} - (-1)^{k+1} \quad \text{as we wished.}
\end{align*}
\]

This is **not** the induction that we used before. Here we need to go back **one step and two steps**. Sometimes we need to go back even further. This is called **strong induction**.

**Strong induction.**

- **Base of strong induction.** \( P(n_0) \) is true.

- **Inductive step.** For any integer \( k \geq n_0 \), \( P(i) \) for \( 1 \leq i \leq k \) \( \Rightarrow \) \( P(k+1) \) is true.

Then For any integer \( n \geq n_0 \), \( P(n) \) is true.

**Ex.** Any integer \( n \geq 2 \) can be written as product of primes.
Definition. An integer \( n \geq 2 \) is called prime if the only positive divisors of \( n \) are 1 and \( n \).

Recall. For any non-zero integers \( a \) and \( b \), \( a | b \iff |a| \leq |b| \).

* For any positive numbers \( x \) and \( y \), \( \min \{x,y\} \leq \sqrt{xy} \).

If \( n \) is NOT prime, then it has a positive divisor \( d \) other than 1 and \( n \). So

\[
\begin{align*}
1 \leq d \leq n & \Rightarrow 1 < d < n \\
& \quad \text{if } d \neq 1, d \neq n \\
& \quad \text{and } n = dk \text{ for some integers } k.
\end{align*}
\]

\( \Rightarrow 0 < k \),

\( \Rightarrow 1 < d \),

\( \Rightarrow k < dk = n \),

\( \Rightarrow 1 < k \).

 Altogether \( n = dk \) for some integers \( 1 < d, k < n \).

And we get the following lemma.

**Lemma.** For any integer \( n \geq 2 \), if \( n \) is NOT prime, then there are integers \( n_1 \) and \( n_2 \) such that

* \( 2 \leq n_1, n_2 \leq n - 1 \)
• \( 2 \leq n_1, n_2 \leq n-1 \)
• \( n = n_1 \cdot n_2 \)

**Remark.** In fact we have \( \min \{ n_1, n_2 \} \leq \sqrt{n} \). So to check if a positive integer is prime or not, it is enough to look at integers \( 1 \leq m \leq \sqrt{n} \).

**Proof of Example.** We use strong induction.

**Base of induction.** \( n = 2 \) is prime. So there is nothing to prove.

(I am using this convention that having a single term is still considered to be a "product", e.g. \( \prod_{i=1}^{n} a_i \) is the product of \( a_i \)'s even if \( n = 1 \).)

**Strong inductive step.** We need to show: for any integer \( k \geq 2 \), any integer \( 2 \leq i \leq k \) can be written as product of primes (strong induction hypothesis.)

\( \Downarrow \)

\( k+1 \) can be written as product of primes.

**Case 1.** If \( k+1 \) is prime, then it is written as product of primes (by our convention).

**Case 2.** If \( k+1 \) is NOT prime, then, by Lemma, there are integers \( k_1 \) and \( k_2 \) such that

• \( 2 \leq k_1, k_2 \leq (k+1)-1 = k \)
\[ 2 \leq k_1, k_2 \leq (k+1)-1 = k \]

\[ k+1 = k_1 \cdot k_2 \]

By the strong induction hypothesis, \( k_1 \) and \( k_2 \) can be written as product of primes. So \( k_1 \cdot k_2 \) can be written as product of primes. And we are done as \( k+1 = k_1 k_2 \).