In the previous lecture we said how we can say $A \subseteq \mathbb{R}$ has a minimum.

Ex. Use quantifiers to say a nonempty subset $A \subseteq \mathbb{R}$ is bounded.

Solution. $\exists \ m, M \in \mathbb{R}, \ \forall a \in A, \ m \leq a \leq M.$

Ex. Prove or disprove that any bounded nonempty subset $A \subseteq \mathbb{R}$ has a minimum.

Solution. We disprove it by proving that $(0,1)$ is bound, but it does not have a minimum.

Bounded. $\forall \ a \in (0,1), \ 0 \leq a \leq 1 \implies$ it is bounded.

No minimum. Suppose to the contrary that it has a min.

$\implies \exists \ x \in (0,1), \ \forall y \in (0,1), \ x \leq y.$ $\otimes$

$x \in (0,1) \implies 0 < x$

$\implies 0 < x/2 \implies 0 < x/2 < x$

$\implies x/2 < x/2 + x/2$

$\implies x/2 \in (0,1) \land x/2 < x$

which contradicts $\otimes.$ $\blacksquare$

In order to get a better understanding of quantifiers and the
- For instance of their order: Let's play.

- Each time a player is supposed to say one of the numbers
  
  1, 2, 3, 4, 5

  A player wins if the mentioned numbers add up to 30.

- After a few rounds we make a conjecture on winning cases and losing cases.

  P: A game is a winning game if the first player has a winning move which changes the game into a losing game.

  N: A game is a losing game if no matter what the first player does, the 2nd player has a winning move.

Using quantifiers:

  P: \( \exists \) a move for player A, \( \forall \) move of player B, A could win.

  N: \( \forall \) move of player A, \( \exists \) move of player B, B could win.

  Alternatively. Game is \( P \Rightarrow \exists \) a move which makes it N.

  Game is \( N \Rightarrow \forall \) move makes it P.

Ex. In a game each player is supposed to say one of

\[ p \]
the numbers 1, 2, 3, 4, 5. A player wins if the numbers add up to \( n \).

Find all the \( n \)'s s.t. the above game is a losing game.

Solution. (In class we have to conjecture what the answer is: \( n \)'s that are multiples of 6.)

We use strong induction to show, for \( n \in \mathbb{Z}^+ \),

\[
G(n) \text{ is } N \iff 6 \mid n.
\]

**Base.** \( G(1) \) is clear \( P \) as the 1st player just says 1.

**Strong inductive step.** For any \( k \in \mathbb{Z}^+ \),

\[
1 \leq m \leq k, \quad 6 \mid m \Rightarrow G(m) \text{ is } N \iff \exists \ y \quad 6 \mid k+1 \Rightarrow G(k+1) \text{ is } N.
\]

\[
6 \mid m \Rightarrow G(m) \text{ is } P \iff 6 \nmid k+1 \Rightarrow G(k+1) \text{ is } P.
\]

**Proof.** \( 6 \mid k+1 \Rightarrow 6 \mid (k+1) - 1 \Rightarrow \) after the first move we get \( G((k+1) - 1) \), which is \( P \) by induction hyp.

\[
\Rightarrow G(k+1) \text{ is } N.
\]

\[
6 \mid k+1 \Rightarrow k+1 = 6q + r \Rightarrow \text{ if the first player takes out the remainder,}
\]

\[
0 \leq r < 6 \quad \text{we get } G(6q) \text{ which is } N.
\]
\[ \Rightarrow \ F(k+1) \text{ is } I. \]