In the previous lecture we said what a winning game \((P)\) and a losing game \((N)\) are.

\[ P: \exists \text{ a move which makes it } N. \]

\[ N: \forall \text{ move we get } P. \]

Let's look at the definition of \underline{limit}. 

In calculus, you have seen \( \lim_{x \to a} f(x) = L \).

Here is its formal definition:

\[ \forall \varepsilon > 0, \exists \delta > 0, 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon. \]

It is like a losing game \underline{for any "move" of the 1st player \((e)\) the second player has a "move" \((s)\).}

Ex: Prove that \( \lim_{x \to 2} x^2 = 4. \)

Proof. \( \forall \varepsilon > 0 \) we need to show that \( \exists \delta > 0 \) s.t.

\[ 0 < |x - 2| < \delta \Rightarrow |x^2 - 4| < \varepsilon. \]

\[ |x^2 - 4| = |x - 2||x + 2| < 5|x - 2| < 5\delta \leq \varepsilon \]
\[ \frac{5}{8} < 1 \Rightarrow |x-2|<1 \Rightarrow |x+2|<5. \]

So \( \delta = \min \frac{\varepsilon}{4}, \frac{\varepsilon}{5} \) is a "good move", i.e.

\[ \forall \varepsilon > 0, \quad 0 < |x-2| < \min \frac{\varepsilon}{4}, \frac{\varepsilon}{5} \Rightarrow |x^2-4| < \varepsilon. \]

Ex. Prove that \( \lim_{x \to 2} \sqrt{x} = \sqrt{2} \).

Proof. \( \forall \varepsilon > 0 \) we need to show that \( \exists \delta > 0 \) s.t.

\[ 0 < |x-2| < \delta \Rightarrow |\sqrt{x} - \sqrt{2}| < \varepsilon. \]

\[
|\sqrt{x} - \sqrt{2}| = |\sqrt{x} - \sqrt{2}| \cdot \frac{|\sqrt{x} + \sqrt{2}|}{|\sqrt{x} + \sqrt{2}|} = \frac{|x-2|}{|\sqrt{x} + \sqrt{2}|} \leq \frac{\delta}{|\sqrt{x} + \sqrt{2}|}
\]

If \( \delta < 1 \) \( \Rightarrow \) \( |x-2| < 1 \)

\[ \Rightarrow 1 < x < 3 \]

\[ \Rightarrow 1 < \sqrt{x} < \sqrt{3} \]

\[ \Rightarrow 1 + \sqrt{2} < \sqrt{x} + \sqrt{2} < \sqrt{3} + \sqrt{2} \]

\[ \Rightarrow \frac{1}{\sqrt{x} + \sqrt{2}} < \frac{1}{1 + \sqrt{2}} < 1 \]

So \( \delta = \min \frac{\varepsilon}{4}, \frac{\varepsilon}{5} \) is a "good move". I.e. we have proved

\[ \forall \varepsilon > 0, \quad 0 < |x-2| < \min \frac{\varepsilon}{4}, \frac{\varepsilon}{5} \Rightarrow |\sqrt{x} - \sqrt{2}| < \varepsilon. \]
The best way to make sure that we have understood this definition is by considering its negation: \( \lim_{x \to a} f(x) \neq L \)

(it is a “winning game”; 1st player has a “winning move” \( \varepsilon_0 \))

\( \exists \varepsilon > 0, \forall \delta > 0, \exists x, 0 < |x - a| < \delta \land |f(x) - L| > \varepsilon_0 \).

More interesting is the case when \( \lim_{x \to a} f(x) \) does NOT exist.

\( \forall \varepsilon > 0, \forall \delta > 0, \exists x, 0 < |x - a| < \delta \land |f(x) - L| > \varepsilon_0 \).

Ex. Prove that \( \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \) does NOT exist.

We will show this on Wednesday.