Today we do one example on induction.

**Theorem.** Let $S_n$ be the ordered finite sequences of positive integers that sum to $n$. For example

$$S_1 = \mathcal{S}(1), \quad S_2 = \mathcal{S}(1, 1), (2),$$

$$S_3 = \mathcal{S}(1, 1, 1), (1, 2), (2, 1), (3).$$

Then the number of elements in $S_n$ is $2^{n-1}$.

In other words, there are $2^{n-1}$ ways to express $n$ as an ordered sum of positive integers.

**Proof.** We use induction on $n$.

**Base.** $|S_1| = 1^1 = 2^1$.

As we have seen, $S_1 = \mathcal{S}(1)$. Hence $|S_1| = 1 = 2^{1-1}$.

**Inductive step.** For any positive integer $k$,

$$|S_k| = 2^{k-1} \implies |S_{k+1}| = 2^k.$$

Assuming that the theorem is true for some natural number $k$; that is, assume that $S_k$ has $2^{k-1}$ elements.

We show that each element of $S_k$ generates two elements of $S_{k+1}$.
Let \((a_1, a_2, \ldots, a_m)\) be an element of \(S_k\). By definition of \(S_k\), we know that \(a_1 + a_2 + \ldots + a_m = k\).

We generate two elements of \(S_{k+1}\) as follows.

The first one is generated by increasing the last term of the sequence by 1 to yield \((a_1, a_2, \ldots, a_{m-1}, a_m+1)\). This is an element of \(S_{k+1}\) since

\[a_1 + \cdots + a_{m-1} + (a_m+1) = a_1 + \cdots + a_m + 1 = k+1.\]

The other element is generated by appending a 1 at the end of the sequence to yield \((a_1, a_2, \ldots, a_m, 1)\). This is also an element of \(S_{k+1}\).

Next we show that each element of \(S_{k+1}\) was generated from one element of \(S_k\). Let \((b_1, \ldots, b_m)\) be an element of \(S_{k+1}\).

- If \(b_m = 1\), then simply eliminate \(b_m\) from the sequence.
- If \(b_m > 1\), then decrease the last entry of the sequence by 1. In this case, \((b_1, \ldots, b_m)\) was generated by

\[
1 \vdash b
\]
$(c_0, c_1, \ldots, c_{m-1}, c_{m-1}) \in \mathbb{Z}_k$ (Note that $b_{m-1}$ is a positive integer because $b_m > 1$.)

Since every element of $S_k$ generates two elements in $S_{k+1}$, $S_{k+1}$ has twice as many elements as $S_k$. By the inductive hypothesis, there are $2^{k-1}$ elements in $S_k$. Thus, there are $2 \times 2^{k-1} = 2^k$ elements in $S_{k+1}$. $\blacksquare$