1. Let \( f(x) = 1 + \frac{1}{x+1} \) for positive real number \( x \).

(a) Prove that \( f(x) \) is increasing, i.e. for any real numbers \( x_1 \) and \( x_2 \),

\[
x_1 \leq x_2 \implies f(x_1) \leq f(x_2).
\]

Let \( a_0 = 1 \), \( a_{n+1} = f(a_n) \).

(b) Prove that for any non-negative integer \( n \),

\[
a_n \leq \frac{1 + \sqrt{5}}{2} =: t_0
\]

[Notice that \( \frac{1}{t_0} = \frac{\sqrt{5} - 1}{2} = t_0 - 1 \). So

\[
f(t_0) = 1 + \frac{1}{1 + \frac{1}{t_0}} = 1 + \frac{1}{t_0} = t_0.
\]

(c) Prove that \( \{a_n\}_{n=0}^{\infty} \) is an increasing sequence, i.e.

for any non-negative integer \( n \), \( a_n \leq a_{n+1} \).

(d) Prove that \( \lim_{n \to \infty} a_n = \frac{1 + \sqrt{5}}{2} \).

[Using a similar technique, one can show that \( b_0 = 2 = 1 + \frac{1}{2} \), \( b_{n+1} = f(b_n) \) defines a decreasing sequence.]
which converges to \( \frac{1 + \sqrt{5}}{2} \). Altogether we have \( 1 + \frac{1}{1 + \frac{1}{1 + \ldots}} = \frac{\sqrt{5} + 1}{2} \).

This is an example of a continued fraction.

2. Prove that for any positive integer \( n \),
\[
1 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.
\]

3. Let \( \sum_{n=0}^{\infty} f_n^2 \) be the Fibonacci sequence, i.e.
\[
f_0 = 0, \quad f_1 = 1, \quad f_{n+1} = f_n + f_{n-1} \quad \text{for any positive integer } n.
\]
Prove that for any positive integer \( n \),
\[
f_n = \frac{1}{\sqrt{\phi}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right].
\]

4. Let \( b_1 = 1, \quad b_{n+1} = 1 + \frac{1}{b_n} \) for any positive integer \( n \).

So we get the following initial terms:
\[
1, \quad 2, \quad \frac{3}{2}, \quad \frac{5}{3}, \quad \frac{8}{5}, \ldots
\]

\( \circ \) Prove that for any positive integer \( n \),
\[
b_n = \frac{f_{n+1}}{f_n},
\]
where \( \sum_{n=0}^{\infty} f_n^2 \) is the Fibonacci sequence.

\( \circ \) Prove that for any positive integer \( n \),
\[
b_{n+1} - b_n = \frac{(-1)^{n+1}}{f_n f_{n+1}}.
\]
\[
\frac{c_{n+1}}{c_n} = \frac{f_n}{f_{n+1}}
\]

[Hint: Use one of the properties of the Fibonacci sequence that I proved in class.]

[Remark: Problem 1 together with 4.a implies that
\[
\lim_{n \to \infty} \frac{f_{2n+1}}{f_{2n}} = \frac{\sqrt{5} + 1}{2}.
\]

Using 4.b we get
\[
\lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \frac{\sqrt{5} + 1}{2}.
\]

\text{You can use Problem 3 to give another proof for this.}]

5. (Postage stamp problem) Prove that any postage greater than 34 can be obtained by stamps of denominations 5 and 9.

[Hint: You need to show for any integer \( n \geq 34 \) there are non-negative integers \( x \) and \( y \) such that
\[
n = 5x + 9y.
\]

1. Use strong induction on \( n \).

2. 34 = 5 \times 5 + 9,

35 = 5 \times 7,

36 = 9 \times 4,

7 = 7.
38 = 5 \times 4 + 9 \times 2