1. (I) Suppose \( a_0, a_1, \ldots, a_n \in [0, 1] \). Prove that

\[
|a_i - a_j| \leq \frac{1}{n}.
\]

(Hint. Use pigeonhole principle, and \( \frac{1}{n} \) at least \( 1 \).)

(II) Let \( \alpha \in \mathbb{R} \). Prove that,

\[
\forall n \in \mathbb{Z}^+, \exists m \in \mathbb{Z}, \exists k \in \mathbb{Z}, \\
0 < m \leq n \land |m\alpha - k| \leq \frac{1}{n}.
\]

(Hint. Let \( a_i = i\alpha - \lfloor i\alpha \rfloor \) and use part (I).)

(III) Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). Prove that for infinitely many pairs of integers \((m,k)\) we have

\[
|\alpha - \frac{k}{m}| \leq \frac{1}{m^2}.
\]

(Hint. Suppose there are only finitely many such pairs:
\((m_1,k_1), \ldots, (m_s,k_s)\). Since \( \alpha \notin \mathbb{Q} \), \( \min\{m_i | \alpha - k_i| | i \leq s\} \neq 0 \).

So for some \( n \in \mathbb{Z}^+ \), \( \frac{1}{n} < \min\{m_i | \alpha - k_i| | i \leq s\} \).

Now use part (II), and notice \( \frac{1}{n} \leq \frac{1}{m} \) if \( \alpha \in \mathbb{Q} \).)

2. (I) Prove that

\[
|x| + |-x| = \begin{cases} 
0 & \text{if } x \in \mathbb{Z}, \\
-1 & \text{if } x \notin \mathbb{Z}.
\end{cases}
\]

(II) Prove that

\[
|2x| = |x| + |x + \frac{1}{2}|.
\]

(Hint. Case 1. \( |2x| \) is even \( \Rightarrow \)
\( \exists k \in \mathbb{Z}, 2k \leq 2x < 2k+1 \Rightarrow k \leq x < k + \frac{1}{2} \land k + \frac{1}{2} \leq x + \frac{1}{2} < k + 1 \).
3. (i) Prove that if $f : A_1 \to A_2$ and $g : B_1 \to B_2$ are bijections, then $h : A_1 \times B_1 \to A_2 \times B_2$, $h(a, b) = (f(a), g(b))$ is a bijection.

(ii) Prove that, if $A_1, \ldots, A_n$ are enumerable sets, then $A_1 \times \cdots \times A_n$ is enumerable.

(Hint. Use induction on $n$, and the fact that we proved in class: $\mathbb{Z}^+ \times \mathbb{Z}^+ \text{ is enumerable}$.)

4. In this exercise you are allowed to use the fact that any positive integer has a unique binary representation, i.e.

$$\forall n \in \mathbb{Z}^+, \exists! m_1, \ldots, m_k \in \mathbb{Z}^{\geq 0}, \quad 0 \leq m_1 < m_2 < \cdots < m_k$$

and

$$n = 2^{m_k} + 2^{m_{k-1}} + \cdots + 2^{m_1}.$$ 

(i) Prove that $\exists X \subseteq \mathbb{Z}^{\geq 0} \mid X \text{ is finite} \implies \mathbb{Z}^{\geq 0}$ is enumerable.

(Hint. Let $f : \exists X \subseteq \mathbb{Z}^{\geq 0} \mid X \text{ is finite} \implies \mathbb{Z}^+$,

$$f(\exists m_1, \ldots, m_k \in \mathbb{Z}^{\geq 0}) = 2^{m_1} + \cdots + 2^{m_k}.$$)

(ii) Prove that there is no surjection

$$g : \exists X \subseteq \mathbb{Z}^{\geq 0} \mid X \text{ is finite} \implies \mathcal{P}(\mathbb{Z}^{\geq 0}),$$

where $\mathcal{P}(\mathbb{Z}^{\geq 0})$ is the power set of $\mathbb{Z}^{\geq 0}$.

(Hint. Cantor.)
5. Determine if the following functions are injective or surjective.

Justify your answers.

(I) \( f: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}, \ f((a, b)) = 3a - 2b \).

(II) Let \( A \subseteq X \), and \( \lambda: \mathcal{P}(X) \rightarrow \mathcal{P}(X), \ \lambda(B) = A \Delta B \).

(Hint: What is \( \lambda \circ \lambda(B) \)?)

(III) Let \( Y \) be a non-empty subset of \( X \), and

\( \eta: \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \ \eta(B) = Y \cap B \).

6. Suppose \( f: X \rightarrow X \) is a function and \( f \circ f = f \).

Prove that, \( \forall x \in X, \ x \in \text{Im}(f) \iff f(x) = x \).