1. Prove that for any integer \( n \) one and exactly one of the numbers \( n \) and \( n+1 \) is even.

(b) Prove that, for any integer \( n \),

\[ n(n+1) \] is even.

Solution (a) \( n \) is even \( \implies n = 2k \) for some integer \( k \)

\[ \implies n+1 = 2k+1 \]

\[ \implies n+1 \] is odd (we proved in class)

\( n \) is odd \( \implies n = 2k+1 \) for some integer \( k \)

\[ \implies n+1 = 2k+2 = 2(k+1) \]

\[ \implies n+1 \] is even as \( k+1 \) is integer.

(b) Case-by-Case

Case 1. \( n \) is even.

\[ 2 \mid n \implies n = 2k \] for some integer

\[ \implies n(n+1) = 2k(n+1) \]

\[ \implies 2 \mid n(n+1) \] as \( k(n+1) \) is integer.

Case 2. \( n \) is odd.

\[ n \] is odd \( \implies n+1 \] is even \( \implies n+1 = 2k' \) for some (part a)

\[ \implies n(n+1) = 2k'n \]

\[ \implies 2 \mid n(n+1) \] as \( k'n \) is integer.

Second Proof. We prove by contradiction. Suppose to the contrary that \( n(n+1) \) is odd for some integer \( n \).

\( \implies n \) and \( n+1 \) are odd (in class we proved \( mn \) is odd \( \iff m \) and \( n \) are odd.)
which contradicts part (a).

2. Prove that $201x - 9y = 2$ has no integer solution.
   Solution. Suppose to the contrary that it has integer solutions $x_0, y_0$.
   \[ 2 = 201x - 9y_0 = 3(67x - 3y_0) \quad \Rightarrow \quad 3 \mid 2 \quad \text{as} \quad 67x_0 - 3y_0 \] is integer \[ \Rightarrow \exists 2 < 3 \quad \text{which is a contradiction} \quad \text{(in class we proved)} \]
   For non-zero integers $a, b$, $a \mid b \Rightarrow |a| \leq |b|$. \]

3. Prove that for any positive real numbers $x, y, z$
   \[ \frac{\sqrt{x^2 + y^2 + z^2}}{3} \geq \frac{x + y + z}{3} \]
   Proof. Backward argument
   \[ \frac{\sqrt{x^2 + y^2 + z^2}}{3} \geq \frac{x + y + z}{3} \iff \frac{x^2 + y^2 + z^2}{3} \geq \left(\frac{x + y + z}{3}\right)^2 
   \]
   \[ \iff \frac{x^2 + y^2 + z^2}{3} \geq \frac{x^2 + y^2 + z^2 + 2xy + 2xz + 2yz}{9} 
   \]
   \[ \iff 9x^2 + 9y^2 + 9z^2 \geq x^2 + y^2 + z^2 + 2(xy + xz + yz) 
   \]
   \[ \iff 2(x^2 + y^2 + z^2) \geq 2(xy + xz + yz) 
   \]
   \[ \iff x^2 + y^2 + z^2 \geq xy + xz + yz \quad \text{(is proved in class.)} \]

4. Determine if the following statements are true or not.
   Justify your answer.
   (a) For any integers $m$ and $n$,
   \[ 6 \mid mn \iff 6 \mid m \lor 6 \mid n \]
   (b) For any integers $m$ and $n$,
(b) For any integers \( m \) and \( n \),
\[ 6 \mid m \lor 6 \mid n \implies 6 \mid mn. \]

(c) For any integers \( m \) and \( n \),
\[ 3 \mid mn \implies 3 \mid m \lor 3 \mid n. \]

Solution. (a) False; let \( m = 3 \) and \( n = 2 \).
Then \( 6 \mid (3)(2) \) and \( 6 \nmid 3 \) and \( 6 \nmid 2 \).

(If \( 6 \mid 3 \), then \( 6 \leq 3 \) which is a contradiction.
If \( 6 \mid 2 \), then \( 6 \leq 2 \).)

(b) Case-by-case.

Case 1. \( 6 \mid m \).
\[ 6 \mid m \implies m = 6k \text{ for some integer } k \]
\[ \implies mn = 6kn \]
\[ \implies 6 \mid mn \text{ as } kn \text{ is an integer.} \]

Case 2. \( 6 \mid n \)

By a similar argument as in case 1, we have \( 6 \mid mn \).

(c) We prove by contradiction. Suppose to the contrary that for some integers \( m \) and \( n \) we have
\[ 3 \mid mn, \ 3 \mid m, \ 3 \mid n. \]
So by the hint there are integers \( k, l \) s.t. \( m = 3k + 1 \) and \( n = 3l + 1 \).

Hence
\[ mn = (3k + 1)(3l + 1) \]
\[ = 9kl + 3l + 3k + 1 \]
\[ = 3(3kl + l + k + 1). \]
\[ = 3(3k \pm l \pm k) \pm 1. \]  
\[ 3 \mid mn \Rightarrow mn = 3k' \text{ for some integer } k'. \]  
\[ \text{(i), (ii) } \Rightarrow 5k' - 3(3kl \pm l \pm k) = \pm 1. \]  
\[ \Rightarrow \pm 3(k' - kl \pm l \pm k) = 1 \]  
\[ \Rightarrow 3 \mid 1 \Rightarrow 3 \leq 1 \text{ which is a contra.} \]  

5. Let \( d \) be an integer more than 1, and \( a_1, a_2, b_1, \) and \( b_2 \) are integers. Suppose \( d \mid a_1 - a_2 \) and \( d \mid b_1 - b_2. \) 

Prove that \( d \mid (a_1 + b_1) - (a_2 + b_2) \) and \( d \mid a_1 b_1 - a_2 b_2. \)

**Proof.** \( d \mid a_1 - a_2 \Rightarrow \) for some integer \( k, \)

\[ a_1 - a_2 = dk \]  
\[ d \mid b_1 - b_2 \Rightarrow \) for some integer \( l, \)

\[ b_1 - b_2 = dl. \]

\[ (a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) + (b_1 - b_2) \]
\[ = dk + dl \]  
\[ = d(k + l) \]
\[ \Rightarrow d \mid (a_1 + b_1) - (a_2 + b_2) \text{ as } k + l \text{ is an integer.} \]

\[ a_1 b_1 - a_2 b_2 = (a_1 - a_2)b_1 + a_2(b_1 - b_2) \]
\[ = dk b_1 + a_2 dl \]  
\[ = d(k b_1 + a_2 l) \]
\[ d \mid a_1 b_1 - a_2 b_2 \text{ as } kb_1 + a_2 l \text{ is an integer.} \]