1. Let \( f: (0, 1] \to [-1, 1] \), for any \( k \in \mathbb{Z}^{\geq 0} \),
\[
f(x) = (-1)^k \quad \text{if} \quad \frac{1}{2^{k+1}} < x \leq \frac{1}{2^k}.
\]
So its graph looks like:

![Graph of function](image)

Prove that \( \lim_{x \to 0^+} f(x) \) does not exist.

Proof: We have to show \( \forall \varepsilon > 0, \exists \delta > 0, \exists x, \quad 0 < x < \delta \land |f(x) - L| \geq \varepsilon. \) \( \forall \varepsilon > 0, \exists x, \quad 0 < x < \delta \land |f(x) - L| \geq \varepsilon. \) \( \quad \star \)

We claim that in fact any \( 0 < \varepsilon < 1 \), e.g. \( \varepsilon = \frac{1}{2} \), satisfies \( \star \). \( \quad \exists \varepsilon > 0, \quad \exists \delta > 0, \quad 0 < x < \delta \land |f(x) - L| \geq \frac{1}{2}. \) \( \quad \boxdot \)

To prove \( \boxdot \), first we show it is enough to prove the following:
\[
\forall \varepsilon > 0, \exists \delta > 0, \quad 0 < x_1, x_2 < \delta \land f(x_1) = 1 \land f(x_2) = -1. \quad \Box
\]

Suppose \( \Box \) holds. Then, for any \( L \in \mathbb{R} \) and \( \delta > 0 \), let \( x_1 \) and \( x_2 \) be the numbers that satisfy \( \Box \). Then we claim either \( x_1 \) satisfies \( \boxdot \) or \( x_2 \) does.

If not, then \( |f(x_1) - L| < \frac{1}{2} \) and \( |f(x_2) - L| < \frac{1}{2} \).
\[
\Rightarrow \left\{ \begin{array}{l}
|1 - L| < \frac{1}{2} \quad \Rightarrow \quad L > \frac{1}{2}, \quad \text{which is a contradiction.}
\end{array} \right.
\]

\[
\left\{ \begin{array}{l}
|\frac{1}{2} - L| < \frac{1}{2} \quad \Rightarrow \quad L < \frac{1}{2}.
\end{array} \right.
\]
Proof of (i) Suppose \( k \) is a positive integer such that \( \frac{1}{2^k} < \delta \). Then \( 0 < \frac{1}{2^{k+1}} < \frac{1}{2^k} < \delta \) and \( f(\frac{1}{2^k}) = (\epsilon)^k \) and \( f(\frac{1}{2^{k+1}}) = (\epsilon)^{k+1} \). So \( \frac{1}{2^k} \) and \( \frac{1}{2^{k+1}} \) can be \( x_1 \) and \( x_2 \) (or \( x_2 \) and \( x_1 \)).

Notice that \( \frac{1}{2^k} < \delta \iff \frac{1}{\delta} < 2^k \iff \log_2(1/\delta) < k \).

So it is enough to take an integer \( k > \log_2(1/\delta) \).

2. (a) Prove or disprove: \( \forall x \in \mathbb{R}, (\forall \epsilon > 0, |x| \leq \epsilon) \implies x = 0 \).

(b) Prove or disprove: \( \forall x \in \mathbb{R}, \forall \epsilon > 0, (|x| \leq \epsilon \implies x = 0) \).

Solution. (a) It is true, and we prove it by contradiction. Suppose to the contrary that its negation holds:

\( \exists x \in \mathbb{R}, (\forall \epsilon > 0, |x| \leq \epsilon) \land x \neq 0 \),

which is the same as:

\( \exists x \in \mathbb{R} \setminus \{0\}, \forall \epsilon > 0, |x| \leq \epsilon \). \( \Box \)

Notice that, for \( x \in \mathbb{R} \setminus \{0\} \), \( 0 < |x| \). So \( 0 < lx/2 < |x| \), which contradicts \( \Box \).

(a) An alternative, fairly close, method is proving the contrapositive statement: \( \forall x \in \mathbb{R}, (x \neq 0 \implies (\exists \epsilon > 0, |x| > \epsilon)) \)

\( \equiv \forall x \in \mathbb{R} \setminus \{0\}, \exists \epsilon > 0, |x| > \epsilon \). \( \therefore \)

For a given \( 0 < x \in \mathbb{R} \), we need to find \( \epsilon > 0 \) which satisfies \( \therefore \). As above, we know \( \epsilon = lx/2 \) satisfies \( \therefore \).

(b) It is false. We show that its negation is true.

\( \exists x \in \mathbb{R}, \exists \epsilon > 0, |x| < \epsilon \land x \neq 0 \). \( \Box \)

\( \Box \) is clearly true. For instance, \( x = 1, \epsilon = 2 \) satisfy \( \Box \).
3. Prove that \( A \times (B \cup C) = (A\times B) \cup (A \times C) \).

**Proof.** \((x,y) \in A \times (B \cup C) \iff x \in A \land y \in B \cup C \)
\[ \iff x \in A \land (y \in B \lor y \in C) \]
\[ \iff (x \in A \land y \in B) \lor (x \in A \land y \in C) \]
\[ \iff (x,y) \in A \times B \lor (x,y) \in A \times C \]
\[ \iff (x,y) \in (A \times B) \cup (A \times C). \]

4. (a) Find all possible \( a \in \mathbb{R} \) such that

\[ \exists x \in \mathbb{R}, \ x^2 - 2x + a^2 = 0. \]

(b) Find all possible \( a \in \mathbb{R} \) such that

\[ \exists ! x \in \mathbb{R}, \ x^2 - 2x + a^2 = 0. \]

**Solution.**
(a) We have to find all \( a \in \mathbb{R} \) such that \( x^2 - 2x + a^2 = 0 \) 

has a real-valued solution.

\[ x^2 - 2x + a^2 = 0 \iff x^2 - 2x + 1 = 1 - a^2 \]
\[ \iff (x-1)^2 = 1 - a^2 \]

It has a real-valued solution if and only if \( 1 - a^2 \geq 0 \)
\[ \iff a^2 \leq 1 \iff |a| \leq 1 \iff -1 \leq a \leq 1. \]

(b) \( x^2 - 2x + a^2 = 0 \) has a unique real-valued solution if and only if \( (x-1)^2 = 1 - a^2 \) has a unique real-valued solution.

If \( 1 - a^2 < 0 \), it has no solution. If \( 1 - a^2 > 0 \), it has two solutions. So \( 1 - a^2 = 0 \), which implies \( a = \pm 1 \).

5. Prove that there are \( 2^n \) functions \( f : \{1, 2, \ldots, n\} \rightarrow \{0, 1\} \).

**Proof.** We use induction on \( n \).

**Base.** \( n = 1 \). There are 2 functions \( f_1, f_2 : \{1\} \rightarrow \{0, 1\} \).

\( f_1(1) = 0 \) and \( f_2(1) = 1 \). And \( 2 = 2^1 \).

**Inductive step.** For any positive integer \( k \),
number of functions \( f: \mathbb{Z}_1, \ldots, k \rightarrow \mathbb{Z}_0, \mathbb{Z}_1 \) is \( 2^k \),

number of functions \( f: \mathbb{Z}_1, \ldots, k+1 \rightarrow \mathbb{Z}_0, \mathbb{Z}_1 \) is \( 2^{k+1} \).

For any function \( f: \mathbb{Z}_1, \ldots, k+1 \rightarrow \mathbb{Z}_0, \mathbb{Z}_1 \), we have that either \( f(k+1) = 0 \) or \( f(k+1) = 1 \).

How many functions \( f: \mathbb{Z}_1, \ldots, k+1 \rightarrow \mathbb{Z}_0, \mathbb{Z}_1 \) are there such that \( f(k+1) = 0 \) ?

Any such function is uniquely determined by its values on \( \mathbb{Z}_1, \ldots, k \), and vice versa any function \( g: \mathbb{Z}_1, \ldots, k \rightarrow \mathbb{Z}_0, \mathbb{Z}_1 \) can be uniquely extended to a function
\[
f: \mathbb{Z}_1, \ldots, k, k+1 \rightarrow \mathbb{Z}_0, \mathbb{Z}_1
\]
such that \( f(k+1) = 0 \) (to be precise \( f(i) = g(i) \) for \( 1 \leq i \leq k \)). Hence the number of such functions by the induction hypothesis is \( 2^k \).

Similarly the number of functions \( f: \mathbb{Z}_1, \ldots, k+1 \rightarrow \mathbb{Z}_0, \mathbb{Z}_1 \) such that \( f(k+1) = 1 \) is \( 2^k \).

Hence the number of functions \( f: \mathbb{Z}_1, \ldots, k+1 \rightarrow \mathbb{Z}_0, \mathbb{Z}_1 \) is \( 2^k + 2^k = 2^{k+1} \).

6. For \( A \subseteq X \), the characteristic function \( 1_A \) of \( A \) is
\[
1_A : \mathbb{Z}_0, \mathbb{Z}_1 \rightarrow \mathbb{Z}_0, \mathbb{Z}_1, \quad 1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \in \mathbb{Z}_0, \mathbb{Z}_1 \setminus A \end{cases}
\]

(a) Prove that
\[
1_{A \cap B} = 1_A \cdot 1_B
\]

(b) Prove that
\[
1_A^c + 1_A = 1_X
\]

\( (a) \) Proof. \[ \begin{array}{cccccccc}
\times \in A & \times \in B & \times \in A \cap B & 1_A(\times) & 1_B(\times) & 1_A(\times) \cdot 1_B(\times) & 1_{A \cap B}(\times) \\
T & T & T & 1 & 1 & 1 & 1 \\
T & T & 1 & 1 & 1 & 1 & 1 \\
T & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array} \]
(a) Proof.

<table>
<thead>
<tr>
<th>$x \in A$</th>
<th>$x \notin A$</th>
<th>$x \in B$</th>
<th>$x \notin B$</th>
<th>$1_A(\alpha)$</th>
<th>$1_B(\alpha)$</th>
<th>$1_{A \cup B}(\alpha)$</th>
<th>$1_A(\alpha) \cdot 1_B(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

So $1_A(\alpha) \cdot 1_B(\alpha) = 1_{A \cap B}(\alpha)$ for any $\alpha$.

(b) Proof.

<table>
<thead>
<tr>
<th>$x \in A$</th>
<th>$x \in A^c$</th>
<th>$1_A(\alpha)$</th>
<th>$1_A(\alpha)$</th>
<th>$1_A(\alpha) + 1_A(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

$1_A(\alpha) + 1_A(\alpha) = 1 = 1_X(\alpha)$ for any $\alpha \in X$.

**Corollary** $1_{A \cup B} = 1_A + 1_B - 1_A \cdot 1_B$.

**Proof** (Method 1)

$1_{A \cup B} = 1 - 1_{(A \cup B)^c}$ (part (b))

$= 1 - 1_{A^c \cap B^c}$

$= 1 - 1_A^c \cdot 1_B^c$ (part (a))

$= 1 - (1 - 1_A)(1 - 1_B)$ (part (b))

$= 1 - (1 - 1_A - 1_B + 1_A \cdot 1_B)$

$= 1_A + 1_B - 1_A \cdot 1_B$.

(Method 2)

Use a truth-table to prove it.

**Exercise**. Can you find $1_{A \Delta B}$ in terms of $1_A$ and $1_B$?

**Remark**. You can see the similarities between Problem 5 and $|P(\{1, 2, \ldots, n\})| = 2^n$. In your next problem set, you will see how $1_A$ can help us to understand the connection between problem 5 and $|P(\{1, \ldots, n\})| = 2^n$ better.
between problem 5 and $|P_{\mathcal{Q}_1, \ldots, n_{\mathcal{Q}}}| = 2^n$ better.