In the previous lecture we proved

**Lemma.** For any integers \(a\) and \(b\),

\[(a \mid b \land b \neq 0) \Rightarrow |a| \leq |b|.

Let's see some of its applications:

**Q.** Does the equation \(14\ m - 49\ n = 1\) have integer solutions? (This type of equations are called Diophantine equations.)

**Solution.** No! Suppose to the contrary that there are integers \(m\) and \(n\) such that

\[14\ m - 49\ n = 1.

Then the left hand side \(14m - 49n = 7(2m - 7n)\) is a multiple of 7 as \(2m - 7n\) is an integer. Hence \(7 \mid 1\). By the above lemma we get

\[|7| \leq |1|,

which is a contradiction. \(\blacksquare\)
The same argument implies.

Lemma. Suppose $a$ and $b$ are two integers.

If $a$ and $b$ have a common divisor $d$ greater than 1, then the equation $ax + by = 1$ has no integer solutions.

Draft/Proof.

\[
\begin{array}{c|c|c}
\text{Given} & \Rightarrow & \text{Goal} \\
\hline
\text{d | a, d | b, d > 1} & \Rightarrow & ax + by = 1 \\
x, y : \text{integer} & & \\
\end{array}
\]

Proof by contradiction

\[
\begin{array}{c|c|c}
\text{Given} & \Rightarrow & \text{Goal} \\
\hline
\text{d | a, d | b, d > 1,} & \Rightarrow & \text{Contradiction} \\
x, y : \text{integer} & & \\
ax + by = 1 & & \\
\end{array}
\]

\[
d \mid a \Rightarrow \text{for some integer } a' \Rightarrow ax + by = da'x + db'y \\
a = da' \\
d \mid b \Rightarrow \text{for some integer } b' \\
b = db' \\
\Rightarrow d \mid ax + by = 1 \\
\text{by lemma} \Rightarrow |d| \leq 1, \\
\text{which is a contradiction.} \blacksquare
\]
In fact, the converse of this lemma is also correct, but it is harder to prove. We will do it later in this course.

Converse of \( P \implies Q \) is \( Q \implies P \). In general \( P \implies Q \) might be true and at the same time \( Q \implies P \) be false.

\[ \text{Biconditional Proposition } \quad P \iff Q = (P \implies Q) \land (Q \implies P). \]

\( P \) if and only if \( Q \).

\( P \) is necessary and sufficient for \( Q \).

\( P \iff Q \) is true exactly when \( P \) and \( Q \) have the same truth value.

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Definition. Let \( n \) be an integer. We say \( n \) is even if \( 2 \mid n \).

We say \( n \) is odd if \( n \) is NOT even.

Important remark. Since the above conditional proposition is defining a phrase, it gets promoted to a biconditional proposition.