In today’s lecture we study some of the properties of the Fibonacci sequence.

\[ F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \quad \text{for any positive integer } n. \]

So it has some similarities with \( a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + a_n} \).

Both of them are recursive, but \( a_{n+1} \) needs only 1 information and \( F_{n+1} \) needs 2.

Let \( v_n = \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} \). So \( v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and

\[ v_{n+1} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}. \]

Hence \( v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( v_{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} v_n \).

Now we have a recursive formula which only depends on 1 step back.

**Theorem.** Let \( A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \). Then for any positive integer \( n \), we have

\[ A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}. \]
Proof. We use induction on n.

Base of induction. For n=1, LHS = \( A^1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \).
RHS = \( \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1+0 & 1 \\ 1 & 0 \end{bmatrix} \).

The induction step. Suppose for a given positive integer k we have \( A^k = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} \). We have to show

\[
A^{k+1} = \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix}.
\]

\[
A^k \cdot A = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F_{k+1} + F_k & F_{k+1} \\ F_k + F_{k-1} & F_k \end{bmatrix} = \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix}.
\]

by the induction hypothesis

Corollary 1. For any positive integer n,

\[
F_{n+1} \cdot F_{n-1} - F_n^2 = (-1)^n
\]

where \( F_0, F_1, \ldots \) is the Fibonacci sequence.

Proof. \( A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \) where \( A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \). So
Hence \[ F_n^2 \cdot F_{n-1} - F_n^2 = \det (A^n) \]
\[ = \det (A)^n = (-1)^n. \] □

**Corollary 2.** For any positive integers \( m, n \), we have
\[ F_{n+m} = F_{m+1} F_n + F_m F_{n-1}, \]
where \( F_0, F_1, \ldots \) is the Fibonacci sequence.

**Proof.** We know that for any matrix \( A \) and positive integers \( m \) and \( n \) we have (why?)
\[ A^{m+n} = A^m \cdot A^n. \]

Let’s use the above equality for \( A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \) and apply the main theorem:
\[
\begin{bmatrix} F_{n+m+1} & F_{n+m} \\ F_{n+m} & F_{n+m-1} \end{bmatrix} = \begin{bmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{bmatrix} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} F_{m+1} F_{n+1} + F_m F_n & F_{m+1} F_n + F_m F_{n-1} \\ F_m F_{n+1} + F_{m-1} F_n & F_m F_n + F_{m-1} F_{n-1} \end{bmatrix}
\]

Comparing the 12-entries, we get \[ F_{n+m} = F_{m+1} F_n + F_m F_{n-1}. \] □
Theorem. For any positive integers $m$ and $n$, we have

$$m | n \Rightarrow F_m | F_n$$

where $F_0, F_1, F_2, \ldots$ is the Fibonacci sequence.

Proof. Let's fix a positive integer $m$. And let $n$ range through multiples of $m$. So $n = mk$ where $k$ is a positive integer.

So we can rewrite what we need to prove:

For a given positive integer $m$,

for any positive integer $k$, $F_m | F_{mk}$.

We use induction on $k$.

Base of induction. $k = 1$. We have to prove $F_m | F_m$,

which is clear as $F_m = F_m \times 1$.

The induction step. Assume for a given positive integer $l$

we have $F_m | F_{ml}$. Now we have to show $F_m | F_{m(l+1)}$.

$$F_m(l+1) = F_{ml+m} = F_{m+1} \cdot F_{ml} + F_m \cdot F_{ml-1}$$

{Corollary 2 applied to $n = ml$ and $m$}
By the induction hypothesis, \( F_m | F_{ml} \). So there is an integer \( s \) such that \( F_{ml} = (F_m)(s) \). So

\[
F_{m(l+1)} = F_{m+1} \cdot F_m \cdot s + F_m \cdot F_{ml-1}
\]

\[
= F_m \left( \frac{F_{m+1} \cdot s + F_{ml-1}}{\text{integer}} \right)
\]

Hence \( F_m | F_{m(l+1)} \).

Remark 1. The converse of the above Theorem is also correct:

\[ F_m | F_n \Rightarrow m | n. \]

Remark 2. A recursive sequence like

\[ x_{n+1} = a \cdot x_n + b \cdot x_{n-1} \]

is called a