René Descartes used coordinates to study geometry. Nowadays we use the idea of _n-tuples_ in many aspects of our life:

**Ex. List of courses:** it has various columns: name, number, location, ...

**List of movies in netflix:** genre, title, length, rating, etc.

**Definition.** Given sets \( X \) and \( Y \), the **Cartesian product** of \( X \) and \( Y \), denoted by \( X \times Y \), is the set

\[
X \times Y = \{ (x, y) \mid x \in X, y \in Y \},
\]

where \( (x, y) \) is an ordered pair, i.e. \( (x_1, y_1) = (x_2, y_2) \) exactly when \( x_1 = x_2 \) and \( y_1 = y_2 \).

Similarly we define \( X_1 \times X_2 \times \cdots \times X_n = \{ (x_1, \ldots, x_n) \mid x_i \in X_i \text{ for } 1 \leq i \leq n \} \), and \( (x_1, \ldots, x_n) = (x_1', \ldots, x_n') \) if and only if \( x_i = x_i' \) for \( 1 \leq i \leq n \).

**Ex.** Let \( A = \{1, 2\} \) and \( B = \{a, b\} \). List elements of \( A \times B \), and \( B \times A \).

**Solution.** \( A \times B = \{ (1, a), (1, b), (2, a), (2, b) \} \)

\( B \times A = \{ (a, 1), (a, 2), (b, 1), (b, 2) \} \)
Lecture 17: Cartesian product

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We pair each element of $A$ by all the elements of $B$.

In the above example, you can see that $(A \times B) \cap (B \times A) = \emptyset$.

Ex. Let $A = \{1, 2\}$ and $B = \{1, 3, 4\}$. Find $(A \times B) \cap (B \times A)$.

Solution

$A \times B = \{(1, 1), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4)\}$

$B \times A = \{(1, 1), (1, 2), (3, 1), (3, 2), (4, 1), (4, 2)\}$

$(A \times B) \cap (B \times A) = \{(1, 1)\}$.

Lemma. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Proof. $(x,y) \in (A \times B) \cap (C \times D) \iff (x,y) \in A \times B \land (x,y) \in C \times D$

\[ \iff x \in A \land y \in B \land x \in C \land y \in D \]

\[ \iff (x \in A \land x \in C) \land (y \in B \land y \in D) \]

\[ \iff x \in A \cap C \land y \in B \cap D \]

\[ \iff (x,y) \in (A \cap C) \times (B \cap D) \]

Warning. $(A \times B) \cup (C \times D)$ is not necessarily equal to $(A \cup C) \times (B \cup D)$. (why?)
Based on your intuition of cardinality of finite sets, you can see that $|A \times B| = |A| \cdot |B|$ if $A$ and $B$ are finite sets.

Ex. In the following pictures in how many ways can we go from $X$ to $Z$ by passing $Y$ only once.

Solution. We can "label" each path with an element of $\mathbb{S}1,2,3 \times \mathbb{S}a,b$. And any element of $\mathbb{S}1,2,3 \times \mathbb{S}a,b$ is a label of a path. So there is a "matching" (the technical term is bijection as we will learn later) between the possible paths and elements of $\mathbb{S}1,2,3 \times \mathbb{S}a,b$.

So there are 6 possible paths. ■

The key point in the above example is the following:

We often count objects by finding a bijection between them and a more familiar set. A set whose cardinality is already known.
“Definition” A function carries three pieces of information:

. Two sets: one is called domain and the other is called codomain.

. A rule: assigns a unique element of codomain to each element of domain

We either write \( f: X \rightarrow Y \) and then specify its rule, or

\[
\begin{align*}
X & \rightarrow Y \\
\ast & \mapsto f(x)
\end{align*}
\]

. You have worked with a lot of functions in calculus, but in an inaccurate way. In the following examples we will see some of these inaccuracies.

Ex. Is the following a function?

\[ f: \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = \frac{1}{x}. \]

Answer. No, \( f \) is NOT defined at \( 0 \).

By changing its domain, we can address this issue:

\[ f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}, \quad f(x) = \frac{1}{x} \quad \text{is a function.} \]
Ex. Is the following a function?

\[ f: \mathbb{R} \rightarrow \mathbb{R}^+, \quad f(x) = x^2. \]

**Answer.** No, it is NOT. It assigns 0 to 0 which does not belong to the codomain \( \mathbb{R}^+ \).

By changing the codomain we can address this issue:

\[ f: \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = x^2 \text{ is a function.} \]

Ex. Is the following a function?

\[ f: \mathbb{R}^+ \rightarrow \mathbb{R}, \quad f(x) = y \text{ if } y^2 = x. \]

**Answer.** No, it is NOT. This rule does NOT assign a unique element of codomain to, let’s say, 1. We have \((\pm 1)^2 = 1\).

Changing the codomain can resolve this issue:

\( (f: \mathbb{R}^+ \rightarrow \mathbb{R}^+, \quad f(x) = y \text{ if } y^2 = x ) \) is a function.

In fact, in this case, \( f: \mathbb{R}^+ \rightarrow \mathbb{R}^+, \quad f(x) = \sqrt{x} \).

**Composition of functions** Let \( X \xrightarrow{f} Y \) and \( Y \xrightarrow{g} Z \) be two functions; suppose codomain of \( f \) is equal to the domain...
of \( g \). Then we can form a new function called the composition of \( f \) and \( g \), denoted by \( g \circ f \).

Domain of \( g \circ f = \text{Domain of } f \)

Codomain of \( g \circ f = \text{codomain of } g \)

Rule of \( g \circ f : \ x \mapsto g(f(x)) \).

**Ex.** Let \( f : \mathbb{R}\setminus\{0\} \to \mathbb{R}, \ f(x) = \frac{1}{x} \). Find \( f \circ f \).

**Answer.** It does **not** make sense to talk about \( f \circ f \). A codomain of \( f \) is not equal to the domain of \( f \). This issue can be resolved by changing the codomain of \( f \).

Let \( f : \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}, \ f(x) = \frac{1}{x} \). Then

\[
(f \circ f)(x) = f(f(x)) = \frac{1}{f(x)} = \frac{1}{\frac{1}{x}} = x.
\]

**Remark.** \( f \circ f \) is not equal to \( I : \mathbb{R} \to \mathbb{R}, \ I(x) = x \) as they have different (co)domains.