In the previous lecture we defined an invertible function:

We say \( X \xrightarrow{f} Y \) is invertible if

1. it has a left inverse: \( \exists \ Y \xrightarrow{g} X, \ g \circ f = I_X \)
2. it has a right inverse: \( \exists \ Y \xrightarrow{h} X, \ f \circ h = I_Y \)

**Theorem.** Suppose \( X \xrightarrow{f} Y \) is a function.

\( f \) is invertible if and only if \( f \) is bijective.

We start by proving two lemmas:

**Lemma 1.** Suppose \( X \xrightarrow{f} Y \) is a function.

\( f \) has a left inverse if and only if \( f \) is injective.

**Proof.** \((\Rightarrow)\) \( \exists \ Y \xrightarrow{g} X, \ g \circ f = I_X \) which is injective.

Hence, by a theorem that we proved in the previous lecture, \( f \) is injective.

\((\Leftarrow)\) Suppose \( X \xrightarrow{f} Y \) is injective. We would like to define a function \( Y \xrightarrow{g} X \) such that \( g \circ f = I_X \), which means, for any \( x \in X \), \( g(f(x)) = x \).
Lecture 21: Injection and having a left inverse

This means $g$ should undo $f$ on the image of $f$ and can be anything outside of $\text{Im}(f)$.

Here is a formal definition:

Choose $x_0 \in X$ (we can do that since $X \neq \emptyset$). Define $Y \xrightarrow{g} X$ as follows:

$$g(y) = \begin{cases} x & \text{if } y = f(x) \text{ for some } x \in X \\ x_0 & \text{if } y \in Y \setminus \text{Im}(f) \end{cases}$$

We need to show $g$ is a function (we say $g$ is well-defined).

[Recall that to show "an assigning rule" defines a function from $X$ to $Y$, we have to check three things:

1. This "rule" can be applied to all the elements of $X$.

2. This "rule" assigns elements of $Y$ to any element of $X$.

3. This "rule" assigns a unique element of $Y$ to any element of $X$.

For instance, we have seen that $f: \mathbb{R}^+ \rightarrow \mathbb{R}$, $f(x) = y$ if $y^2 = x$ does NOT define a function. This rule assigns two elements of $\mathbb{R}$ to 1. Both 1 and -1 are assigned to 1.]
And then we have to check that $g_f^* = I_X$. 
well-definedness of $g$. It clearly assigns elements of $Y$ to any element of $X$. We have to check why it assigns a unique element:

1. If $y \in Y \setminus \text{Im}(f)$, then $x_0$ is assigned to $y$ with no ambiguity.
2. Suppose $y \in \text{Im}(f)$, and $x_1$ and $x_2$ can be assigned to $y$. So $f(x_1) = y \land f(x_2) = y$, which implies $f(x_1) = f(x_2)$. Since $f$ is injective and $f(x_1) = f(x_2)$, we get that $x_1 = x_2$. So a unique element of $X$ is assigned to $y$.

Checking $g \circ f = I_X$.

Both $g \circ f$ and $I_X$ are functions from $X$ to $X$. So we have to check only that $(g \circ f)(x) = I_X(x)$ for any $x \in X$.

$$(g \circ f)(x) = g(f(x)) = g(y) \quad \text{where } y = f(x)$$

$$= x \quad \text{the way we defined } g.$$  

$$= I_X(x).$$

$$\blacksquare$$
Lemma. Suppose \( f : X \rightarrow Y \) is a function.

\( f \) has a right inverse if and only if \( f \) is surjective.

In the proof we will be using an axiom of set theory called **axiom of choice**. First proof will be written and then it will be mentioned where axiom of choice is used.

Proof. \((\rightarrow)\) \( \exists h : Y \rightarrow X \), \( f \circ h = I_Y \). Since \( I_Y \) is surjective, \( f \) is surjective. (In the previous lecture we have proved that \( f \circ f \) is surjective implies that \( f \) is surjective.)

\((\leftarrow)\) we assume \( f \) is surjective. And we have to find \( h : Y \rightarrow X \) such that \( (f \circ h)(y) = y \). So \( h \) should be defined in a way such that \( f(h(y)) = y \).

For any \( y \in Y \), let \( f(y) = \{ x \in X \mid f(x) = y \} \) be the preimage of \( y \). Since \( f \) is surjective, \( f(y) \neq \emptyset \) for any \( y \in Y \).

Let's choose one element of \( f(y) \) and call it \( h(y) \). So we get a function \( h : Y \rightarrow X \) such that \( h(y) \in f(y) \).
So \( f(h(y)) = y \). Hence \( f \circ h = I_Y \) as both of these functions are from \( Y \) to \( Y \) and \( (f \circ h)(y) = f(h(y)) = y = I_Y(y) \).

(Almost)

For a single non-empty set \( Z \), we can get \( z \in Z \). But to do it simultaneously for a family of non-empty sets, one needs **axiom of choice**:

Suppose \( F: Y \to P(X) \) be a function such that

\[
\forall y \in Y, \; F(y) \neq \emptyset.
\]

Then there is a function \( h: Y \to X \) such that

\[
\forall y \in Y, \; h(y) \in F(y).
\]

Using the axiom of choice for \( F: Y \to P(X) \), \( F(y) = \{f(y)\} \)

we get the desired \( h: Y \to X \).
Proof of Theorem. ($f$ is invertible $\iff$ $f$ is bijective.)

$f$ is invertible $\iff$ $f$ has a left inverse $\iff$ $f$ is injective  
(by Lemma 1).

$f$ has a right inverse $\iff$ $f$ is surjective  
(by Lemma 2).

$f$ is injective and surjective $\iff$ $f$ is bijective.

($\Leftarrow$) $f$ is bijective $\implies$ $f$ is injective and surjective

$f$ is injective $\iff$ $f$ has a left inverse $\iff$ $f$ is invertible  
(by Lemma 1).

$f$ is surjective $\iff$ $f$ has a right inverse  
(by Lemma 2).