In the previous lecture we defined the notion of **equipotent** sets:

\[ A \sim B \iff \text{There exists a bijection } A \xrightarrow{f} B. \]

We have discussed:

- \( A \cap A \cup A \cap B \Rightarrow B \cap A \cup A \cap C \).
- For two non-empty finite sets \( A \) and \( B \),

\[ A \cap B \iff |A|=|B|. \]

In particular, if \( B \) is finite, \( A \subseteq B \), and \( A \cap B \),

then \( A = B \).

**Q** What if \( B \) is NOT finite?

**Ex. (Hilbert's hotel)** \( \mathbb{Z}^+ \sim \mathbb{Z}^0 \).

(In Hilbert's hotel, we have room 1, room 2, ... (infinitely many rooms). All of them are occupied. Let's say guest number \( i \) is in the \( i \)th room. A new guest arrives; let's call her guest number 0. Can we make a room available for her?)

**Proof.** We have to construct a bijection \( f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^0 \).

Let \( f(k) = k-1 \) for any \( k \in \mathbb{Z}^+ \). It is easy to see that \( f \) is a bijection. For instance, you
can check that \( g: \mathbb{Z}^0 \to \mathbb{Z}^+, \ g(k) = k+1 \) is an inverse of \( f \). \[ \blacksquare \]

**Definition.** A set \( X \) is called **enumerable** if \( X \sim \mathbb{Z}^+ \).

A set \( X \) is called **countable** if \( X \) is either finite or it is enumerable.

**Ex.** \( \mathbb{Z} \) is enumerable.

**Proof.** "We have to enumerate elements of \( \mathbb{Z} \)."

\[ \ldots -7 -5 -3 -1 \ 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ldots \]

This picture suggests the following functions:

\[ f: \mathbb{Z}^+ \to \mathbb{Z}, \ f(n) = \begin{cases} -k & \text{if } n = 2k+1, \\ k & \text{if } n = 2k. \end{cases} \]

and

\[ g: \mathbb{Z} \to \mathbb{Z}^+, \ g(n) = \begin{cases} 2n & n > 0, \\ -2n+1 & n \leq 0. \end{cases} \]

Check that \( f \) is well-defined and \( f \) is an inverse of \( g \). \[ \blacksquare \]
Though writing the details of the above proof might be a bit tricky, the whole idea is in the mentioned "labeling" or "enumerating" of elements of \( \mathbb{Z} \): 

To show a set \( A \) is enumerable it is enough to present a method of labelling \( A \)'s elements by numbers 1, 2, 3, \ldots in a way that for sure all the elements of \( A \) get labelled at some point. (and only once).

Ex. \( \mathbb{Z}^+ \times \mathbb{Z}^+ \) is enumerable.

Proof. Clearly this red path passes through all the points of \( \mathbb{Z}^+ \times \mathbb{Z}^+ \) once and exactly once. So we get a bijection between \( \mathbb{Z}^+ \times \mathbb{Z}^+ \) and \( \mathbb{Z}^+ \).

Lemma. \((A_1 \sim A_2 \text{ and } B_1 \sim B_2) \Rightarrow A_1 \times B_1 \sim A_2 \times B_2\) for any non-empty sets \( A_1, A_2, B_1, \text{ and } B_2 \).
Proof. $A_1 \cup A_2 \Rightarrow \exists A_1 \xrightarrow{f} A_2$ which is bijective, and

$B_1 \cup B_2 \Rightarrow \exists B_1 \xrightarrow{g} B_2$ which is bijective.

Let $A_1 \times B_1 \xrightarrow{h} A_2 \times B_2$, $h(a_1, b_1) = (f(a_1), g(b_1))$. Since $f$ and $g$ are bijective, they have inverses $f^{-1}$ and $g^{-1}$. Now let $A_2 \times B_2 \xrightarrow{h'} A_1 \times B_1$, $h'(a_2, b_2) = (f^{-1}(a_2), g^{-1}(b_2))$.

Then $(h \circ h')(a_2, b_2) = h(f^{-1}(a_2), g^{-1}(b_2))$

$= (f(f^{-1}(a_2)), g(g^{-1}(b_2)))$

$= (a_2, b_2)$,

and similarly you can see $(h' \circ h)(a_1, b_1) = (a_1, b_1)$. So $h'$ is an inverse of $h$. Hence $h$ is a bijection, which implies

$A_1 \times B_1 \sim A_2 \times B_2$. ■

Corollary. If $A$ and $B$ are enumerable, then $A \times B$ is enumerable.

Proof. $A$ and $B$ are enumerable $\Rightarrow \exists A \sim \mathbb{Z}^+$ $\Rightarrow A \times B \sim \mathbb{Z}^+ \times \mathbb{Z}^+$

$\left[ \begin{array}{c} B \sim \mathbb{Z}^+ \\ \end{array} \right]$ (by Lemma).

$\mathbb{Z}^+ \times \mathbb{Z}^+ \sim \mathbb{Z}^+$. So $A \times B \sim \mathbb{Z}^+$, and so $A \times B$ is enumerable. ■