HOMEWORK 1 SOLUTIONS

MICHELLE BODNAR

Note: I will freely use the logical equivalences proved in the lecture notes.

Problem 1

For this problem you should set up a truth table for each statement. Two statements are logically equivalent if and only if their columns are identical in a truth table.

Problem 2

$$\begin{aligned}(P \vee Q) \implies R &\equiv \neg(P \vee Q) \vee R \\ &\equiv (\neg P \wedge \neg Q) \vee R \\ &\equiv (\neg P \vee R) \wedge (\neg Q \vee R) \\ &\equiv (P \implies R) \wedge (Q \implies R)\end{aligned}$$

Alternately, we can break the proof into cases:
Case 1: $P \vee Q$ is true. Then the left hand side is true if and only if R is true. For the right hand side, if R is true then both implications are true, so the entire statement is. If R is false then either $P \implies R$ or $Q \implies R$ is false since at least one of P and Q is true. Thus, the entire right hand side is false.

Case 2: $P \vee Q$ is false. Then the left hand side is always true. Further, both P and Q are false which means that the statements $P \implies R$ and $Q \implies R$ must both be true, ensuring that the right hand side is always true as well.

Problem 3

Suppose P is true and Q is false. Then $P \implies Q$ is false but $Q \implies P$ is true. Thus, the statements cannot be logically equivalent.

Problem 4

(a) Recall the definition of absolute value:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}.$$

If $a \geq 0$ then $|a| = a$, so $|a| \geq a$. If $a < 0$ then multiplying both sides by -1 gives $0 < -a = |a|$.
(b) If $b \geq 0$ then $|b| = b$ so $|b|^2 = b^2$. If $b < 0$ then $|b| = -b$ so $|b|^2 = (-b)(-b) = (-1)^2 b^2 = b^2$.

(c) If $c+d \geq 0$ then:

$$\begin{aligned} |c+d| &= c+d \\ &\leq |c| + d \text{ by part (a)} \\ &\leq |c| + |d| \text{ by part (a).} \end{aligned}$$

If $c+d < 0$ then:

$$\begin{aligned} |c+d| &= -(c+d) \\ &= -c + -d \\ &\leq |-c| + -d \\ &\leq |-c| + |-d| \\ &= |c| + |d|. \end{aligned}$$

Problem 5

$$\begin{aligned} P \implies (Q \vee R) &\equiv \neg P \vee (Q \vee R) \\ &\equiv (\neg P \vee Q) \vee R \\ &\equiv \neg(\neg P \vee Q) \implies R \\ &\equiv (P \wedge (\neg Q)) \implies R \text{ this proves the first equivalence} \\ &\equiv ((P \wedge (\neg Q)) \wedge \neg R) \implies \bot \\ &\equiv (P \wedge (\neg Q) \wedge \neg R) \implies \bot \end{aligned}$$

Problem 6

Since $d | m_1 - m_2$ and $d | n_1 - n_2$, there exist integers k and ℓ such that $dk = m_1 - m_2$ and $d\ell = n_1 - n_2$. Then we have:

$$\begin{aligned} (m_1 + n_1) - (m_2 + n_2) &= m_1 - m_2 + n_1 - n_2 \\ &= dk + d\ell \\ &= d(k + \ell) \end{aligned}$$

so we conclude $d | ((m_1 + n_1) - (m_2 + n_2))$. Furthermore, we have $dkn_1 = (m_1 - m_2)n_1$ and $d\ell m_2 = (n_1 - n_2)m_2$. Adding these two equations together yields

$$m_1 n_1 - m_2 n_2 = dkn_1 + d\ell m_2 = d(kn_1 + \ell m_2).$$

Therefore $d | (m_1 n_1 - m_2 n_2)$.