Note: I will freely use the logical equivalences proved in the lecture notes.

Problem 1

For this problem you should set up a truth table for each statement. Two statements are logically equivalent if and only if their columns are identical in a truth table.

Problem 2

\[(P \lor Q) \implies R \equiv \neg (P \lor Q) \lor R \equiv (\neg P \land \neg Q) \lor R \equiv (\neg P \lor R) \land (\neg Q \lor R) \equiv (P \implies R) \land (Q \implies R)\]

Alternately, we can break the proof into cases:
Case 1: \(P \lor Q\) is true. Then the left hand side is true if and only if \(R\) is true. For the right hand side, if \(R\) is true then both implications are true, so the entire statement is. If \(R\) is false then either \(P \implies R\) or \(Q \implies R\) is false since at least one of \(P\) and \(Q\) is true. Thus, the entire right hand side is false.

Case 2: \(P \lor Q\) is false. Then the left hand side is always true. Further, both \(P\) and \(Q\) are false which means that the statements \(P \implies R\) and \(Q \implies R\) must both be true, ensuring that the right hand side is always true as well.

Problem 3

Suppose \(P\) is true and \(Q\) is false. Then \(P \implies Q\) is false but \(Q \implies P\) is true. Thus, the statements cannot be logically equivalent.

Problem 4

(a) Recall the definition of absolute value:
\[|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}\]

If \(a \geq 0\) then \(|a| = a\), so \(|a| \geq a\). If \(a < 0\) then multiplying both sides by \(-1\) gives \(0 < -a = |a|\).

(b) If \(b \geq 0\) then \(|b| = b\) so \(|b|^2 = b^2\). If \(b < 0\) then \(|b| = -b\) so \(|b|^2 = (-b)(-b) = (-1)^2b^2 = b^2\).
(c) If \( c + d \geq 0 \) then:

\[
|c + d| = c + d \\
\leq |c| + d \quad \text{by part (a)} \\
\leq |c| + |d| \quad \text{by part (a)}.
\]

If \( c + d < 0 \) then:

\[
|c + d| = -(c + d) \\
= -c - d \\
\leq |c| + -d \\
\leq |c| + |d|.
\]

**Problem 5**

\[
P \implies (Q \vee R) \equiv \neg P \vee (Q \vee R) \\
\equiv (\neg P \vee Q) \vee R \\
\equiv \neg(\neg P \vee Q) \implies R \\
\equiv (P \land (\neg Q)) \implies R \text{ this proves the first equivalence} \\
\equiv ((P \land (\neg Q)) \land \neg R) \implies \bot \\
\equiv (P \land (\neg Q) \land \neg R) \implies \bot
\]

**Problem 6**

Since \( d|m_1 - m_2 \) and \( d|n_1 - n_2 \), there exist integers \( k \) and \( \ell \) such that \( dk = m_1 - m_2 \) and \( d\ell = n_1 - n_2 \). Then we have:

\[
(m_1 + n_1) - (m_2 + n_2) = m_1 - m_2 + n_1 - n_2 \\
= dk + d\ell \\
= d(k + \ell)
\]

so we conclude \( d|(m_1 + n_1) - (m_2 + n_2) \). Furthermore, we have \( dkn_1 = (m_1 - m_2)n_1 \) and \( d\ell m_2 = (n_1 - n_2)m_2 \). Adding these two equations together yields

\[
m_1n_1 - m_2n_2 = dkn_1 + d\ell m_2 = d(kn_1 + \ell m_2).
\]

Therefore \( d|(m_1n_1 - m_2n_2) \).