HOMEWORK 3 SOLUTIONS

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Problem 1

(a) Let x and y be positive real number. Then we have

$$\begin{aligned} x < y &\implies \frac{1}{y} \leq \frac{1}{x} \\ &\implies 1 + \frac{1}{y} \leq 1 + \frac{1}{x} \\ &\implies \frac{1}{1+\frac{1}{x}} \leq \frac{1}{1+\frac{1}{y}} \\ &\implies f(x) \leq f(y) \end{aligned}$$

so we conclude that the function is increasing.

(b) We'll proceed by induction. For the base case we have $a_0 = 1 = \frac{1}{2} + \frac{1}{2} \leq \frac{1}{2} + \frac{\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}$. Now suppose that $a_k \leq \frac{1+\sqrt{5}}{2}$ for some integer $k \geq 0$. Since f is increasing, we have

$$a_{k+1} = f(a_k) \leq f\left(\frac{1+\sqrt{5}}{2}\right) = \frac{2\left(\frac{1+\sqrt{5}}{2}\right)+1}{\left(\frac{1+\sqrt{5}}{2}\right)+1} = 2\frac{2+\sqrt{5}}{3+\sqrt{5}}\frac{3-\sqrt{5}}{3-\sqrt{5}} = \frac{1+\sqrt{5}}{2}.$$

By induction, $a_n \leq \frac{1+\sqrt{5}}{2}$ for all $n \geq 0$.

(c) We'll proceed by induction. For the base case, $a_0 = 1$ and $a_1 = 3/2$ so we have $a_0 \leq a_1$. Now suppose that $a_k \leq a_{k+1}$ for some integer $k \geq 0$. Since f is increasing, we have

$$f(a_k) \leq f(a_{k+1})$$

which implies that

$$a_{k+1} \leq a_{k+2}.$$

By induction, we conclude that $a_n \leq a_{n+1}$ for all $n \geq 0$.

(d) Since $\{a_n\}$ is increasing an bounded from above it must converge to a limit L. Then we have

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} f(a_n) = \frac{2L+1}{L+1}.$$

Solving, we see that $L = \frac{1+\sqrt{5}}{2}$ or $L = \frac{1-\sqrt{5}}{2}$. We know to choose the first option since $a_0 = 1$ and a_n is an increasing sequence, so L must be positive. Thus, $L = \frac{1+\sqrt{5}}{2}$. $\square$

Problem 2

We'll proceed by induction. For the base case when $n = 1$ we have $1^2 = 1$ and $\frac{1(1+1)(2+1)}{6} = 1$. Now suppose that $1^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$ for some integer $k \geq 0$. Then we have

$$\begin{aligned}
1^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)((k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

By induction, we conclude that the claim holds for all $n \geq 1$. □

Problem 3

We'll proceed by induction. When $n = 1$ we have $b_1 = 1$ and $\frac{F_2}{F_1} = 1/1 = 1$. Now suppose $b_n = \frac{F_{k+1}}{F_k}$ for some integer $k \geq 1$. Then we have

$$\begin{aligned}
b_{k+1} &= 1 + \frac{1}{b_k} \\
&= 1 + \frac{F_k}{F_{k+1}} \\
&= \frac{F_{k+1} + F_k}{F_{k+1}} \\
&= \frac{F_{k+2}}{F_{k+1}}
\end{aligned}$$

as desired. By induction, the claim holds for all $n \geq 1$. □

Problem 4

The hint shows how to write n as a positive integer linear combination of 5 and 9 for $34 \leq n \leq 38$. In other words, we can make 34 through 38 cent postage using 5 and 9 cent stamps. The proof will proceed by strong induction. Suppose that for some integer $k \geq 38$ we can create all postage amounts $34 \leq j \leq k$. We now need to show that we can make $k+1$ cent postage. By assumption, $k + 1 - 5 = k - 4 \geq 38 - 4 = 34$. By the induction hypothesis, we can make $k - 4$ cent postage using 5 and 9 cent stamps. Adding a single 5 cent stamp to this yields the desired $k + 1$ cent postage. By strong induction, we conclude that all postage amounts greater equal 34 are possible using 5 and 9 cent stamps. □

Problem 5

We'll proceed by induction. When $n = 1$, we have $4^1 + 5 = 9 = 3 \cdot 3$, which is divisible by 3. Now suppose that $3|4^k + 5$ for some integer $k \geq 1$. Then we can write $4^k + 5 = 3m$ for some integer m, and we have

$$\begin{aligned} 4^{k+1}+5 &= 4(4^k+5-5)+5 \\ &= 4(3m-5)+5 \\ &= 3(4m)-20+5 \\ &= 3(4m)-15 \\ &= 3(4m-5). \end{aligned}$$

Therefore $3|4^{k+1}+5$. By induction, we conclude $3|4^n+5$ for all positive integers n. □

Problem 6

We'll proceed by induction. It is clear that any 2×2 square grid with one square removed can be covered by a single L-shaped tile. Now suppose that any $2^k\times 2^k$ square grid with one square removed can be covered by L-shaped tiles for some integer $k\geq 1$. Suppose we're given a $2^{k+1}\times 2^{k+1}$ square grid G with a single square removed. Break this into 4 $2^k\times 2^k$ square grids G_1, G_2, G_3, and G_4 by cutting G vertically down the middle and horizontally across the middle. By assumption, exactly one of these has one square removed. Without loss of generality, we may assume it is G_1. By the induction hypothesis, we can cover G_1 with L-shaped pieces. Using a single L-shaped piece, cover the 3 center squares which come from G_2, G_3, and G_4. By the induction hypothesis, we can now tile the remainder of G_2, G_3 and G_4 with L-shaped pieces, thereby coving all of G. By induction, we conclude that any $2^n\times 2^n$ square grid with one square removed can be covered by L-shaped tiles. □

Problem 7

We'll proceed by strong induction. By assumption, $x^1+\frac{1}{x^1}$ is an integer, so the claim holds in the base case when $n=1$. Now suppose that for some integer $k\geq 1$ we have $x^m+\frac{1}{x^m}$ is an integer for all $1\leq m\leq k$. Observe that

$$\left(x^k+\frac{1}{x^k}\right)\left(x+\frac{1}{x}\right)=x^{k+1}+\frac{1}{x^{k+1}}+x^{k-1}+\frac{1}{x^{k-1}}.$$

By the induction hypothesis, $x^k+\frac{1}{x^k}=r_1$ for some integer r_1, $x+\frac{1}{x}=r_2$ for some integer r_2, and $x^{k-1}+\frac{1}{x^{k-1}}=r_3$ for some integer r_3. Thus, $x^{k+1}+\frac{1}{x^{k+1}}=r_1r_2-r_3$ which is an integer. By induction we conclude that $x^n+\frac{1}{x^n}$ for all positive integers n. □