HOMEWORK 8 SOLUTIONS

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Problem 1

Suppose there exists a surjection $f : A \to B$. Recall that $|B| \le |A|$ if and only if there exists an injection from B to A. We'll construct one presently. Define a function $g : B \to A$ as follows: For each $b \in B$, we know there exists at least one $a \in A$ such that $f(a) = b$. Set $g(b)$ equal to one such a. (You can refresh your memory about this sort of thing by looking back over the Axiom of Choice lecture notes.) Suppose $a = g(b_1) = g(b_2)$ for some $b_1, b_2 \in B$. By definition of g, we must have $f(a) = b_1$ and $f(a) = b_2$, so $b_1 = b_2$. Therefore g is an injection, so $|B| \le |A|$.

Now suppose $|B| \le |A|$. Then there exists an injection $g : B \to A$. We need to construct a function $f : A \to B$ which is surjective. Define f as follows: If $x \in Im(g)$, there exists a unique $y \in Y$ such that $g(y) = x$. In this case set $f(x) = y$. Otherwise, let y_0 be some fixed element of Y. For each $x \in X \backslash Im(g)$, set $f(x) = y_0$. Then f is clearly a surjection since for each $y \in Y$ we have $f(g(y)) = y$.

Problem 2

Define a function $f : (a, b) \to (0, 1)$ as follows:

$$f(x) = (b - a)x + a.$$

This function is a bijection since we can write down its inverse: $f^{-1} : (a, b) \to (0, 1)$, $f^{-1}(y) = \frac{y-a}{b-a}$.

Problem 3

(a) Let $f(x) = \arctan(x)$. Then f is a bijection from $(-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$.

(b) By problem 2, $(0, 1) \sim (-\frac{\pi}{2}, \frac{\pi}{2})$. By part (a), $(-\frac{\pi}{2}, \frac{\pi}{2}) \sim \mathbb{R}$. Thus, $(0, 1) \sim \mathbb{R}$.

Problem 4

Suppose $|A| = |B|$. Then there exists a bijection $f : A \to B$. Define a function $g : P(A) \to P(B)$ by $g(X) = \{f(x) | x \in X\}$. I claim g is a bijection. To see that this is a bijection, it is enough to write down an inverse. Define $h : P(B) \to P(A)$ by $h(Y) = \{f^{-1}(y) | y \in Y\}$. This definition makes sense because f is a bijection, so f^{-1} actually exists. For any $X \in P(A)$ we have

$$h(f(X)) = h(\{f(x)|x \in X\}) = \{f^{-1}(f(x))|x \in X\} = \{x|x \in X\} = X.$$

Similarly, you can check $f(h(Y)) = Y$ for all $Y \in P(B)$. Therefore g is invertible so it is a bijection.

Problem 5

(a) Define $f : \{X \subseteq \mathbb{Z}_{\ge 0} | X \text{ is finite}\} \to \mathbb{Z}^+$ as in the hint, by

$$f(\{m_1, \ldots, m_k\}) = 2^{m_1} + \cdots + 2^{m_k}.$$

To see that f is surjective, let $n \in Z^+$. Then n has a binary representation $n = 2^{i_1} + \cdots + 2^{i_j}$ where $0 \le i_1 < \cdots < i_j$ and $f(\{i_1, \ldots, i_j\}) = n$. Furthermore, f is injective because the binary representation of a number is unique. In other words, if $2^{m_1} + \cdots + 2^{m_k} = 2^{i_1} + \cdots + 2^{i_j}$ then $k = j$ and $m_\ell = i_\ell$ for each $1 \le \ell \le k$. Thus, f is a bijection.

(b) Suppose toward a contradiction that there exists a surjection

$$g : \{X \subseteq \mathbb{Z}_{\ge 0} | X \text{ is finite}\} \to P(\mathbb{Z}_{\ge 0}).$$

Let f be defined as in part (a). Define a function $h : Z_{\ge 0} \to \{X \subseteq \mathbb{Z}_{\ge 0} | X \text{ is finite}\}$ by $h(n) = f^{-1}(n+1)$. Then h is a bijection since it is a composition of bijections. However, this means that $g \circ h : \mathbb{Z}_{\ge 0} \to P(\mathbb{Z}_{\ge 0})$ is a surjection, a contradiction to Cantor's theorem.

Problem 6

(a) Not injective, since $f(0,0) = f(2,3)$. However, f is surjective. Let $n \in \mathbb{Z}$ be arbitrary. If n is even, $n = 2k$ for some integer k and we have $f(0, -k) = 2k = n$. If n is odd then $n = 2k+1$ for some integer k. Then $f(1, 1-k) = 3 - 2(1-k) = 2k+1 = n$. Therefore f is surjective.

(b) Observe that

$$\ell \circ \ell(B) = \ell(A \Delta B) = A \Delta (A \Delta B) = (A \Delta A) \Delta B = \varnothing \Delta B = B.$$

Thus, $\ell \circ \ell = I_{P(X)}$ so ℓ is both a left and right inverse of iteself. Thus, ℓ is a bijection, so it is both injective and surjective.

(c) If $Y = X$ then $B \cap Y = B \cap X = B$ so that π is just the identity function. In this case, π is certainly a bijection. Now suppose that $Y \ne X$. Then there exists some $x \in X$ such that $x \notin Y$. Then we have $\pi(\varnothing) = \varnothing = \pi(\{x\})$, so π fails to be injective. However, π is surjective because for any $C \in P(Y)$ we have $\pi(C) = C \cap Y = C$.

Problem 7

(a) By a previous homework assignment, we know that $|A|$ is even if and only if $|A \Delta \{1\}|$ is odd. Thus ℓ_1 and ℓ_2 are indeed well-defined. In particular, the symmetric difference operator is a well-defined function and the functions map each element of their respective domains to their respective codomains.

(b) Let $A \in X_O$. Then we have

$$\ell_1 \circ \ell_2(A) = \ell_1(A \Delta \{1\}) = (A \Delta \{1\}) \Delta \{1\} = A \Delta (\{1\} \Delta \{1\}) = A \Delta \varnothing = A.$$

Furthermore, for any $A \in X_E$ we have

$$\ell_2 \circ \ell_1(A) = \ell_2(A \Delta \{1\}) = (A \Delta \{1\}) \Delta \{1\} = A \Delta (\{1\} \Delta \{1\}) = A \Delta \varnothing = A.$$

We see that ℓ_1 is both a left and right inverse of ℓ_2, so it is the unique inverse of ℓ_2.

(c) We know that $X_E \cup X_O = P(\{1, \ldots, n\})$ and $X_E \cap X_O = \varnothing$. We also have that $|P(\{1, \ldots, n\})| = 2^n$. Furthermore, $|X_E| = |X_O|$ since ℓ_2 is a bijection between them. Thus, by the previous homework,

$$2^n = |P(\{1, \ldots, n\})| = |X_E| + |X_O| - |X_E \cap X_O| = 2|X_E| - |\varnothing| = 2|X_E|.$$

Therefore $|X_E| = 2^{n-1} = |X_O|$.