Lecture 05: Induction and the Fibonacci sequence

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In today's lecture we study some of the properties of the Fibonacci sequence in order to further understand proof by induction. Let

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \text{ for every positive integer } n.$$

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. We are going to compute A^n and find its connection with the Fibonacci sequence.

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^3 = A \cdot A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

To compute the next powers faster, let's verbalize what multiplication by A (from left) does to a matrix $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} (0)(x)+(1)(z) & (0)(y)+(1)(t) \\ (1)(x)+(1)(z) & (1)(y)+(1)(t) \end{bmatrix} = \begin{bmatrix} z & t \\ x+z & y+t \end{bmatrix}$$

So the bottom row is moved to the first row, and the rows are added to compute the 2nd row.

$$A^4 = A \cdot A^3 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1+2 & 2+3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix},$$

$$A^5 = A \cdot A^4 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 2+3 & 3+5 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix}.$$

Based on the above computations we can feel the connection

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with the Fibonacci sequence and formulate the following conjecture:

$A^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}$ for every positive integer n.

<u>Theorem</u>. Suppose $F_0, F_1, \ldots$ is the Fibonacci sequence and

$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Then, for every positive integer n,

$$A^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}. \qquad (*)$$

<u>Proof</u>. We use induction on n.

<u>Base of induction.</u> We have to show that $(*)$ holds for $n=1$.

The left hand side is $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, the right hand side is

$\begin{bmatrix} F_0 & F_1 \\ F_1 & F_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$, and the base of induction follows.

<u>Induction step.</u> Suppose for a positive integer k,

$$A^k = \begin{bmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{bmatrix}. \quad \text{(Induction hypothesis)}$$

We have to show that

$$A^{k+1} = \begin{bmatrix} F_k & F_{k+1} \\ F_{k+1} & F_{k+2} \end{bmatrix}.$$

$$A^{k+1} = A \cdot A^k \underset{\text{by the induction hypothesis}}{=} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{bmatrix} = \begin{bmatrix} F_k & F_{k+1} \\ F_k + F_{k-1} & F_{k+1} + F_k \end{bmatrix}$$

$$= \begin{bmatrix} F_k & F_{k+1} \\ F_{k+1} & F_{k+2} \end{bmatrix} \qquad \blacksquare$$

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<u>Corollary</u> For every positive integer n,

$$F_{n+1} \cdot F_{n-1} - F_n^2 = (-1)^n$$

where $F_0, F_1, \ldots$ is the Fibonacci sequence.

<u>Proof.</u> By the previous theorem,

$$A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix},$$

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. So

$$\det(A^n) = \det \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}. \qquad (*)$$

Because $\det(XY) = \det(X)\det(Y)$ for every two 2-by-2 matrices X and Y, $\det(A^n) = \det(A)^n$ for every positive integer n. Hence, by $(*)$, we obtain

$$\det(A)^n = F_{n+1} \cdot F_{n-1} - F_n^2.$$

Since $\det A = \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = -1$, we obtain that

$$(-1)^n = F_{n+1} \cdot F_{n-1} - F_n^2. \qquad \blacksquare$$

Notice that for every positive integers m and n, we have

$$A^m \cdot A^n = \underbrace{\underbrace{(A \cdot \cdots \cdot A)}_{m \text{ times}} \cdot \underbrace{(A \cdot \cdots \cdot A)}_{n \text{ times}}}_{m+n \text{ times}} = A^{m+n}.$$ By the previous theorem,

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we have

$$A^m = \begin{bmatrix} F_{m-1} & F_m \\ F_m & F_{m+1} \end{bmatrix}, \quad A^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}, \text{ and } A^{m+n} = \begin{bmatrix} F_{m+n-1} & F_{m+n} \\ F_{m+n} & F_{m+n+1} \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} F_{m-1} & F_m \\ F_m & F_{m+1} \end{bmatrix} \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{m+n-1} & F_{m+n} \\ F_{m+n} & F_{m+n+1} \end{bmatrix}.$$

Comparing the (2,1) entries, we conclude that

$$F_m F_{n-1} + F_{m+1} F_n = F_{m+n}.$$

Altogether we have the following result.

<u>Corollary</u> For every positive integers m and n,

$$F_{m+n} = F_m F_{n-1} + F_{m+1} F_n.$$

where $F_0, F_1, \ldots$ is the Fibonacci sequence.

The following is an interesting property of Fibonacci sequences.

<u>Theorem</u>. Suppose m and n are positive integers. Then, $m \mid n$ implies that $F_m \mid F_n$ where $F_0, F_1, \ldots$ is the Fibonacci sequence.

<u>Proof</u>. Notice that $m \mid n$ exactly when $n = md$ for some integer d.

Since $m, n > 0$, $d > 0$. Hence it is enough to show that for every positive integer d, $F_m \mid F_{md}$ (where m is a fixed positive integer). We use induction on d.

Base of induction. $d=1$. We have to prove $F_m \mid F_m$, which is clear as $F_m = F_m \times 1$.

The induction step. Suppose for a positive integer k, $F_m \mid F_{mk}$.

induction hypothesis

We have to show that $F_m \mid F_{m(k+1)}$.

$$F_{m(k+1)} = F_{mk+m}$$

$$= F_{mk} F_{m-1} + F_{mk+1} F_m$$

By the previous corollary,

$F_{r+s} = F_r F_{s-1} + F_{r+1} F_s$

let $r = mk$ and $s = m$

By the induction hypothesis, $F_m \mid F_{mk}$. This means

$$F_{mk} = F_m \cdot a \quad \text{for some integer } a.$$

Hence, $F_{m(k+1)} = F_m \cdot a \cdot F_{m-1} + F_{mk+1} F_m$

$$= F_m \underbrace{(a \cdot F_{m-1} + F_{mk+1})}_{\text{is an integer}}.$$

Thus $F_m \mid F_{m(k+1)}$. ∎

Notice that $F_3 = 2$ implies F_{3k} is even for every positive integer k, and $F_4 = 3$ implies $3 \mid F_{4k}$ for every $k \in \mathbb{Z}^+$.

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These statements are not trivial, either. Here are two other remarks:

Remark 1. The converse of the above Theorem is also correct:

$$F_m \mid F_n \Longrightarrow m \mid n.$$

Later in the course, we discuss Euclid's algorithm to find the greatest common divisor of two integers. That can be used to show $\gcd(F_m, F_n) = F_{\gcd(m,n)}$. This, in particular, implies the mentioned converse proposition.

Remark 2. A sequence which is defined as follows

$$x_{n+1} = a\, x_n + b\, x_{n-1}$$

is called a linear recursive sequence. And if $x_0 = 0$, $x_1 = 1$, then similar statements as above hold for x_n.

Lecture 05: Strong induction

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So far to find F_{50}, we need to compute all $F_0, F_1, \dots, F_{49}$.

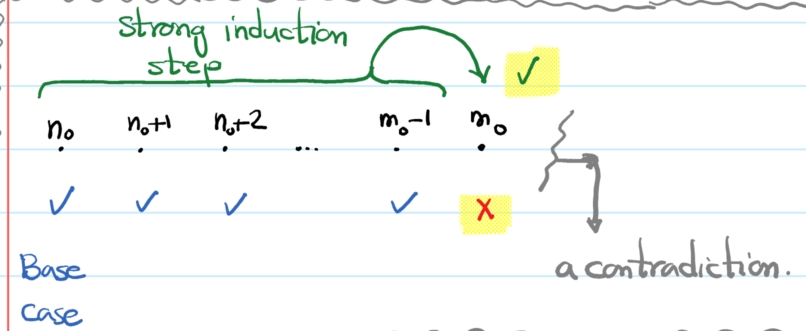
Can we compute F_{50} directly (or at least can we say how large it is?)

To do so, we have to work with a stronger form of the induction principle that we call the strong induction principle. In the induction step, we only use the previous step to go to the next. In the strong induction, we go to the next step using <u>all</u> the previous steps.

<u>Strong induction principle</u> To prove, for every integer $\underline{n \geq n_0}$, $P(n)$ holds, it is enough to show

(Base of strong induction) $P(n_0)$ holds.

(Strong induction step) Suppose for an integer $k \geq n_0$ the following holds: For every integer $n_0 \leq l \leq k$, $P(l)$ holds. (referred to as the strong induction hypothesis) Then $P(k+1)$ holds.



Suppose $P(n)$ fails for some value. Let m_0 be the smallest number where $P(n)$ fails. Then

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We understand the strong induction better by using it! Our first example is on finding a formula for the n-th term of the Fibonacci sequence.

Before focusing on the Fibonacci sequence, let's consider all the sequences that are given by the same recursive formula: $x_{n+1} = x_n + x_{n-1}$.

Can we find any explicit sequence which satisfy this recursive formula?

Let's examine a sequence of the form $x_n = c^n$ for some complex number c. Can we find c such that $c^{n+1} = c^n + c^{n-1}$ for every positive integer n? Notice that

$$c^{n+1} = c^n + c^{n-1} \iff \frac{c^{n+1}}{c^{n-1}} = \frac{c^n + c^{n-1}}{c^{n-1}}$$

$$\iff c^2 = c+1$$

$$\iff c \text{ is a zero of } x^2 - x - 1 = 0.$$

Let's recall that $x^2 - x - 1 = 0 \iff x^2 - x + \frac{1}{4} = \frac{5}{4}$

$$\iff \left(x - \frac{1}{2}\right)^2 = \frac{5}{4}$$

$$\iff a = \frac{1+\sqrt{5}}{2} \text{ and } b = \frac{1-\sqrt{5}}{2}$$

are the only zeros of $x^2 - x - 1 = 0$.

- $\frac{1+\sqrt{5}}{2}$ is called the golden ratio, and it appears in different places.

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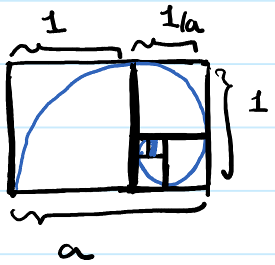
Side note on golden ratio. Since $a^2 = a+1$, we have $a = 1 + \frac{1}{a}$.

So if we start with an a-by-1 rectangle and cut a 1-by-1 square we are left with a 1-by-$1/a$ rectangle, which is similar to the original rectangle. So we can repeat this process. (You might have seen this figure.)



So far we showed that $a^{n+1} = a^n + a^{n-1}$ and $b^{n+1} = b^n + b^{n-1}$ for every positive integer n if $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. Notice that for every numbers x and y, we have

$$(xa^n + yb^n) + (xa^{n-1} + yb^{n-1}) = x(a^n + a^{n-1}) + y(b^n + b^{n-1})$$
$$= x a^{n+1} + y b^{n+1}$$

This means the sequence $xa^0 + yb^0, xa^1 + yb^1, xa^2 + yb^2, \ldots$ satisfies a similar recursive formula as the Fibonacci sequence.

Next we give x and y such that $F_n = xa^n + yb^n$ for every non-negative integer n.

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<u>Theorem</u> (Binet) For every non-negative integer n, we have

$$F_n = \frac{1}{\sqrt{5}}(a^n - b^n), \text{ where } a = \frac{1+\sqrt{5}}{2} \text{ and } b = \frac{1-\sqrt{5}}{2}.$$

<u>Proof</u>. We use strong induction on n.

<u>Base of strong induction</u>. We have to show $F_0 = \frac{1}{\sqrt{5}}(a^0 - b^0)$.

The left hand side is 0 and the right hand side is $\frac{1}{\sqrt{5}}(1-1)=0$.

<u>Strong induction step</u>. Suppose for a non-negative integer k the following holds: for every $0 \le \ell \le k$, $F_\ell = \frac{1}{\sqrt{5}}(a^\ell - b^\ell)$.

We have to show $F_{k+1} = \frac{1}{\sqrt{5}}(a^{k+1} - b^{k+1})$.

We know that $F_{k+1} = F_k + F_{k-1}$ if $k \ge 1$. So we consider two cases.

<u>Case 1</u>. $k=0$.

In this case, we have to show $F_1 = \frac{1}{\sqrt{5}}(a-b)$. The left hand side is $F_1 = 1$, and the right hand side is

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right] = 1.$$

<u>Case 2</u>. $k \ge 1$.

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In this case

$$F_{k+1} = F_k + F_{k-1}$$

$$= \frac{1}{\sqrt{5}}(a^k - b^k) + \frac{1}{\sqrt{5}}(a^{k-1} + b^{k-1})$$ by the strong induction hypothesis

$$= \frac{1}{\sqrt{5}}(a^k - b^k + a^{k-1} - b^{k-1})$$

$$= \frac{1}{\sqrt{5}}\left[(a^k + a^{k-1}) - (b^k + b^{k-1})\right]$$

As we have mentioned earlier, $a^k + a^{k-1} = a^{k+1}$ and $b^k + b^{k-1} = b^{k+1}$

(because a and b are zeros of $x^2 - x - 1 = 0$.) . Hence

$$F_{k+1} = \frac{1}{\sqrt{5}}(a^{k+1} - b^{k+1}). \qquad \blacksquare$$