Lecture 07: Language of set theory

Saturday, October 15, 2016 11:09 PM

In this course we do NOT carefully the axioms of set theory.

We only give a casual introduction to sets and discuss the basics of the language of set theory.

Roughly a set is a well-defined collection of objects. A box containing certain objects.

For instance:
- The set of integer numbers is denoted by $\mathbb{Z}$.
- The set of rational numbers is denoted by $\mathbb{Q}$.
- The set of real numbers is denoted by $\mathbb{R}$.
- The set of complex numbers is denoted by $\mathbb{C}$.

Definition. Objects in a set are called its elements or members.

We write $a \in A$ to say a is an element of A or a is in A.

And we write $a \notin A$ to say $\neg(a \in A)$.

Ex. $1/2 \in \mathbb{Q}$, $1/2 \notin \mathbb{Z}$, $i \in \mathbb{C}$, $i \notin \mathbb{R}$.

A set can be described in different ways.

1. We can list elements of a set.

Lecture 07: Language of set theory

Monday, October 17, 2016 9:29 AM

Ex. $A = \{1, 2\}$.

$1 \in A$, $3 \notin A$, $\{1\} \notin A$.

Ex. $\{\ \}$. This is the empty box

It is called the empty set. It is also denoted by $\emptyset$.

Ex. $B = \{1, \{1, 2\}\}$

$1 \in B$, $2 \notin B$, $\{1\} \notin B$, $\{1, 2\} \in B$

B is a "box" which contains 2 objects: the first object is 1 and the second object is a box which contains 1 and 2.

Ex. $\{\ \} \in \{\{\ \}\}$.

Definition. Two sets A and B are equal if they contain the same collection of members.

"$x \in A \iff x \in B$".

- Reordering elements does NOT change the set.

Ex. $\{1, 1\} = \{1\}$

- Repeating an element does NOT change the set.

Ex. $\{\{1, 2\}, 1\} = \{1, \{1, 2, 1\}, 1\}$.

Ex. $\{\{\ \}\{\ \}\} = \{\emptyset, \{\ \}\}$.

Lecture 07: Language of set theory

Monday, October 17, 2016 1:49 PM

2. Giving the conditions of membership.

<u>Ex.</u>
- $\{n \in \mathbb{Z} \mid 2 \mid n\}$ The set of even numbers
- $\{n \in \mathbb{Z} \mid 2 \nmid n\}$ The set of odd numbers
- $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ The upper-half plane.
- $\{x \in \mathbb{R} \mid x^2 - 1 = 0\} = \{1, -1\}$
- $\{x \in \mathbb{R} \mid x^2 - 1 = 0,\ x \geq 2\} = \varnothing$.

3. Constructing the elements of the set

<u>Ex.</u> $\{2k \mid k \in \mathbb{Z}\} =$ the set of even numbers

$$= \{n \in \mathbb{Z} \mid 2 \mid n\}$$

$$\{2k+1 \mid k \in \mathbb{Z}\} \overset{?}{=} \{n \in \mathbb{Z} \mid 2 \nmid n\}$$

To prove this we have to show for an integer n

$$\left(\begin{array}{l} n \text{ is of the form} \\ 2k+1 \text{ for some} \\ \text{integer } k \end{array}\right) \Longleftrightarrow 2 \nmid n,$$

which we have already proved.

<u>Definition</u>. For two sets A and B, we say A is a subset of B if every element of A is an element of B.

Lecture 07: The language of set theory

Monday, October 17, 2016 2:39 PM

"$x\in A \Rightarrow x\in B$"

Ex. . $\{1\} \subseteq \{1,2\}$

. $\{\{1\}\} \not\subseteq \{1,2\}$ since $\{1\} \in \{\{1\}\} \wedge \{1\} \notin \{1,2\}$.

To show $A \not\subseteq B$ it is enough to find x such that

$$x\in A \wedge x\notin B.$$

. $\{\} \subseteq \{1\}$

In fact $\emptyset \subseteq A$ for every set A.

Ex. Give two sets A and B such that $A\in B \wedge A\subseteq B$.

Solution. $A=\{1\}$ and $B=\{1,\{1\}\}$ are one such example.

$A=\emptyset$ and $B=\{\emptyset\}$ is another.

For two sets A and B, we have

$$A=B \Leftrightarrow (A\subseteq B \wedge B\subseteq A).$$

Ex.. For every set A, we have $A\subseteq A$ and $\emptyset \subseteq A$.

- Using an axiom of set theory we have that, for every set A, $A\notin A$.

- $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Lecture 07: Subsets, cardinality of finite sets

Thursday, October 20, 2016 2:53 PM

"Definition" For a finite set X, the number of elements of X is called the cardinality of X. And it is denoted by $|X|$.

Ex. • $|\{1,1\}| = 1$. • $|\{1,2,\{1,2\}\}| = 3$.

• $|\emptyset| = 0$.

• $|\{\{1\},\{1,1\}\}| = 1$. In this example we are using the fact that $\{1\} = \{1,1\}$, and so $\{\{1\},\{1,1\}\} = \{\{1\}\}$.

Ex. • $|\{1,2,\mathbb{R}\}| = 3$. Elements of this set are 1, 2, and $\mathbb{R}$.

• $|\{1,2,\mathbb{R},\mathbb{C}\}| = 4$.

• $|\{\{-1,1\},\{x\in\mathbb{R} \mid x^2=1\}\}| = 1$.

Here we are using the fact that $\{x\in\mathbb{R} \mid x^2=1\} = \{-1,1\}$.

Definition. For a set X, the set of subsets of X is called its power set, and it is denoted by $P(X)$. So

$$P(X) = \{A \mid A \subseteq X\}.$$

Ex. • $P(\emptyset) = \{\emptyset\}$. So $|P(\emptyset)| = 1$.

• $P(\{1\}) = \{\emptyset, \{1\}\}$. So $|P(\{1\})| = 2$.

Lecture 07: Power set

Thursday, October 20, 2016 3:28 PM

- For every sets A and X, $A \subseteq X \Leftrightarrow A \in \mathcal{P}(X)$.
- For every set X, $\{\emptyset, X\} \subseteq \mathcal{P}(X)$.
- List all the elements of $\mathcal{P}(\{a,b\})$.

To this end, we view a subset as a "club", and we have to decide who can be a member of this club. Say the name of this "club" is A. We have to make two decisions:

① do we want to have $a \in A$? (to have a as a member) ② do we want to have $b \in A$? (to have b as a member)

We can capture all the possible combinations of our decisions in a truth-table.

$a \in A$	$b \in A$	A
T	T	$\{a,b\}$
T	F	$\{a\}$
F	T	$\{b\}$
F	F	$\emptyset$

Hence, $\mathcal{P}(\{a,b\}) = \{\emptyset, \{b\}, \{a\}, \{a,b\}\}$.

Since there are four rows in this truth-table,

$|\mathcal{P}(\{a,b\})| = 4$.

Ex. List all the elements of $\mathcal{P}(\{1, 2, \{1,2\}\})$.

We use a similar strategy as in the solution of the previous example: use a truth-table, this time for propositions:

$1 \in A$, $2 \in A$, and $\{1,2\} \in A$.

Lecture 07: Power set

Thursday, October 20, 2016 3:44 PM

$1 \in A$	$2 \in A$	$\{1,2\} \in A$	A
T	T	T	$\{1,2,\{1,2\}\}$
T	T	F	$\{1,2\}$
T	F	T	$\{1,\{1,2\}\}$
T	F	F	$\{1\}$
F	T	T	$\{2,\{1,2\}\}$
F	T	F	$\{2\}$
F	F	T	$\{\{1,2\}\}$
F	F	F	$\varnothing$

So $\mathcal{P}(\{1,2,\{1,2\}\}) = \{\varnothing, \{\{1,2\}\}, \{2\}, \{2,\{1,2\}\},$
$\{1\}, \{1,\{1,2\}\}, \{1,2\}, \{1,2,\{1,2\}\}\}.$

In particular $|\mathcal{P}(\{1,2,\{1,2\}\})| = 8$.

Using similar ideas we can see that to list elements of $\mathcal{P}(\{a_1, a_2, \ldots, a_n\})$ we have to write a truth-table for n propositions $a_1 \in A, a_2 \in A, \ldots, a_n \in A$. For each i, we have 2 choices: to include a_i in A or not. All together we have $\underbrace{a_1 \in A}_{2 \text{ choices}} \times \underbrace{a_2 \in A}_{2}, \cdots \times \underbrace{a_n \in A}_{2} = 2^n$ possibilities.

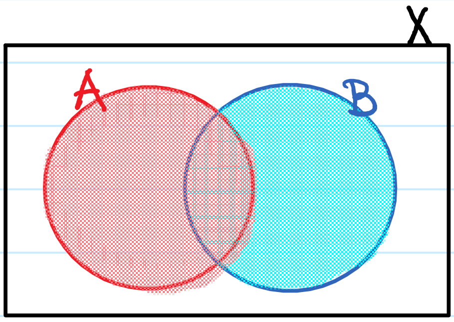
Hence $|\mathcal{P}(X)| = 2^{|X|}$ for a finite set X.

Lecture 07: Set operations

Thursday, October 20, 2016 4:01 PM

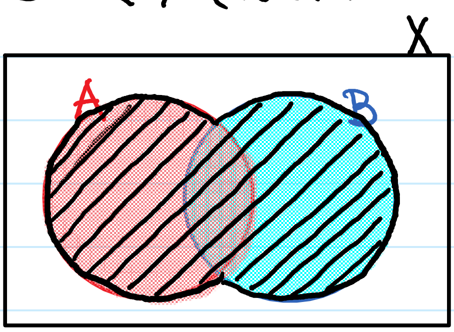
Visualizing concepts often helps us get a better understanding of them, and form an intution. In mathematics, these visualizations are auxilary tools and need to be accompanied by formal proofs. In set theory, we use Venn diagrams to visualize set operations.

Suppose X is a set and $A, B \subseteq X$. The associated Venn diagram looks like:



Thinking about A and B as two "student clubs", we can form a new "club" by merging them. It is called the union of A and B. It is denoted by $A \cup B$. So, for every $x \in X$

$$x \in A \cup B \iff (x \in A \vee x \in B).$$

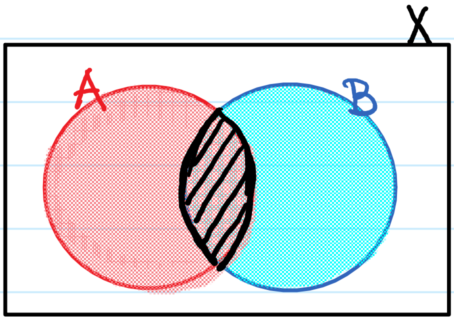


Lecture 07: Set operations

Thursday, October 20, 2016 4:18 PM

Again thinking about A and B as two "student clubs", we can form a new club out of students who are in both of the clubs. It is called the intersection of A and B and it is denoted by $A \cap B$. So, for every $x \in X$,

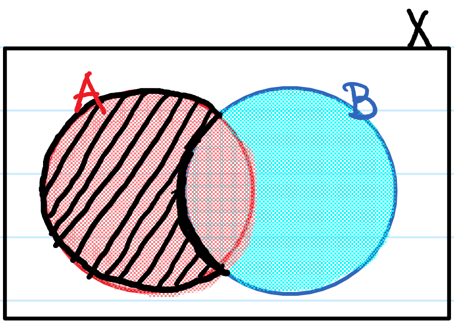
$$x \in A \cap B \iff (x \in A \wedge x \in B).$$



Thinking about A as a "student club" and B as "bad students"(!), we can form a new club out of members who are NOT bad. It is called the set difference of A and B, and it is denoted by either $A \setminus B$ or $A - B$. I will be using $A \setminus B$.

So for every $x \in X$,

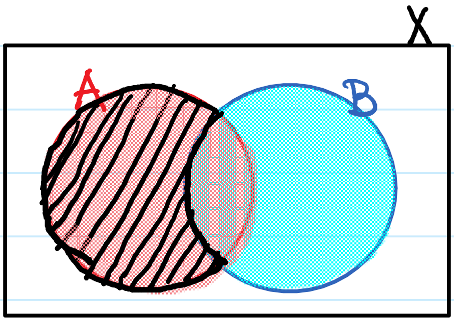
$$x \in A \setminus B \iff (x \in A \wedge x \notin B).$$



Lecture 07: Set operations

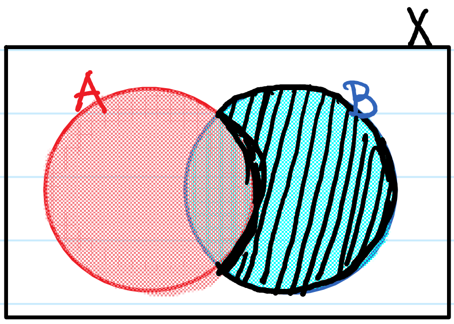
Tuesday, August 16, 2022 10:21 AM

As you can set, the set difference of A and B, $A \setminus B$, is completely different from the set difference of B and A, $B \setminus A$



$A \setminus B$

Students in "math club" who are not "bad".



$B \setminus A$

"Bad" students who are not in "math club".