Lecture 09: Subsets with no minimum

Friday, August 19, 2022 1:40 PM

In the previous lecture we discussed what it means a subset A of real numbers to have a minimum, and what it means for a subset to be bounded:

A non-empty subset A of $\mathbb{R}$ has a minimum if

$$\exists x_0 \in A, \forall y \in A, \quad x_0 \leq y.$$

A non-empty subset A of $\mathbb{R}$ is bounded if

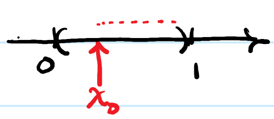
$$\exists m, M \in \mathbb{R}, \forall x \in A, \quad m \leq x \leq M.$$

$\boxed{Q}$ Does every non-empty bounded subset of $\mathbb{R}$ have a minimum?

$\boxed{A}$ No. We show that $(0,1)$ is a bounded non-empty subset of $\mathbb{R}$ which does not have a minimum.

Since $1/2 \in (0,1)$, $(0,1)$ is not empty. Because, for every $x \in (0,1)$, $0 \leq x \leq 1$, $(0,1)$ is bounded.

Next we show that $(0,1)$ does not have a minimum. Suppose to the contrary that $(0,1)$ has a minimum. So $\exists x_0 \in (0,1), \forall y \in (0,1), x_0 \leq y$.



To get a contradiction, it is enough to find an

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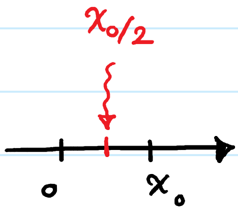
an element of $(0,1)$ which is less than x_0. Intuitively it is clear

that $\frac{x_0}{2}$ should work.



Since $x_0 \in (0,1)$, $0 < x_0$.

So $0 < \frac{x_0}{2}$ and $\frac{x_0}{2} < \frac{x_0}{2} + \frac{x_0}{2} = x_0 < 1$

So $\frac{x_0}{2} \in (0,1) \wedge x_0 \not\leq \frac{x_0}{2}$

which contradics (*) ■

To understand quanifiers better, we learn a bit about game theory. This game is a variant of NIM.

There is a heap of 30 marbles. There are two players Alice (A) and Bob (B). They play in turn. At their turn, they remove $1, 2, \ldots,$ or 6 marbles from the heap. A player wins if s/he picks the last marble. Determine if the first player can win or not.

After we played this game during the lecture, we established certain terminologies. These terminologies are given from the 1st player perspective.

A game is called a winning game (W) if the 1st player has a

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has a strategy to win. A game is called a losing game if the 2nd player has a strategy to win.

1st player	2nd player	game status
A	B	W
B	A	L

$\exists$ a move for the 1st player to turn the game to L.

A game is a winning game means the 1st player has a move to turn the game into a losing game for the other player

1st player	2nd player	game status
A	B	L
B	A	W

$\forall$ move of the 1st player turns the game to W.

A game is a losing game means every move of the 1st player turns the game in to a winning game.

Following two moves for each game, we have:

Winning game. $\exists$ a move for Alice, $\forall$ move of Bob, game is W.

Losing game. $\forall$ move of Alice, $\exists$ a move of Bob, game is L.

You can see the importance of the order of moves, and in an L game, Bob's move depends on Alice's move. For every move of Alice, Bob has to find the right move.

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To understand if the NIM game with 30 marbles is W or L, we start with smaller number of marbles.

If the number of marbles is $1, 2, \dots, 6$, the 1st player can take all the marbles. Hence, these are W.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
L	W	W	W	W	W	W	L	W	W	W	W	W	W	L		

For a heap with 7 marbles, every move of Alice lands on a W. Therefore, this is an L.

Starting with 8, 9, 10, 11, 12, or 13, Alice can move enough marbles to land on 7 which is an L. Hence, all these cases are W.

For a heap with 14 marbles, every move of Alice lands on a W. Therefore, this is an L.

Following this pattern we can conjecture that this is an L exactly when the number of marbles is a multiple of 7. We can prove this by strong induction. We have already discussed the base case. So next we prove the strong induction step. Suppose for a positive integer k, the following holds. For every integer $1 \leq l \leq k$, the mentioned game with l marbles is $\underline{L}$ if and only if 7 divides l. We have to show that the mentioned game with k marbles is $\underline{L}$ if and only if 7 divides k.

($\Leftarrow$) If 7 divides k, $k = 7q$ for some integer q. After every move of Alice, the number of remaining marbles is $7q - r$ for some $1 \leq r \leq 6$. Hence, the number of remaining marbles is not a multiple of 7 and it is less than k. Therefore, by the strong induction hypothesis, it is a $\underline{W}$. This means, in this case, every move of Alice lands on a $\underline{W}$. Hence this case is an $\underline{L}$.

($\Rightarrow$) To show this, we assume k is not a multiple of 7, and we show it is a $\underline{W}$. Since k is not a multiple of 7, by the division algorithm $k = 7q + r$ for some integers q and r, and $1 \leq r \leq 6$. Then Alice takes r marbles, and land on a multiple

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on a multiple of 7. Hence, by the strong induction hypothesis, Alice has a move to land on an <u>L</u>. Therefore, this case is a <u>W</u>.

This finishes the proof of the strong induction step. ∎

Using games, one can naturally see the differences between the quantifiers and the importance of the order of quantifiers.

As the number of quantifiers of a mathematical statement increases, it makes it harder to parse. An important mathematical concept which involves several quantifiers is the notation of limit.

We say $\lim_{x\to a} f(x) = L$ if $f(x)$ gets arbitrarily close to L as x gets closer and closer to a. Let's write a quantitative version of the above qualitative statement. To make sure $f(x)$ is ε-close to L it is enough to make sure x is δ-close to a where the choice of δ depends on ε.

$$\forall \varepsilon>0, \exists \delta>0, \quad 0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon.$$

<u>Definition.</u> We say $\lim_{x\to a} f(x)=L$ precisely when

$$\forall \varepsilon>0, \exists \delta>0, \quad 0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon.$$

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To show $\lim_{x\to a} f(x) = L$, we have to be able to make a good choice for $\delta>0$ depending on the given $\varepsilon>0$ (and of course, a and f).

Ex. Prove that $\lim_{x\to 2} x^2 = 4$.

In the exam, you will be asked a similar question. You should start with the ε-δ- definition of limit.

Proof. We have to show

$$\forall \varepsilon>0, \exists \delta>0, \quad 0<|x-2|<\delta \Rightarrow |x^2-4|<\varepsilon.$$

For a given $\varepsilon>0$, we have to find $\delta>0$ such that the above implication holds. To find a right $\delta>0$, we use backward argument:

$$|x^2-4|<\varepsilon \quad \Leftarrow \quad |x-2||x+2|<\varepsilon \qquad (*)$$

We'd like to reach to this conclusion under the assumption that $|x-2|<\delta$. (We will choose δ to be sufficiently small depending on ε).

In $(*)$, there is an extra factor $|x+2|$ which needs to be controlled.

We have to find an upper bound for this factor. Intuitively, if x is very close to 2, then $|x+2|$ is very close to 4. To make this precise,

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Let's start with an initial "estimate". Let's say we will definitely choose $\delta \le 1$. (The choice of 1 is fairly flexible. Its main point is for us to be able to get an upper bound for $|x+2|$.)

That means we can assume $|x-2|<1$. So $1<x<3$ and $3<x+2<5$, which implies $|x+2|<5$. Hence

$$|x^2-4|<\varepsilon \Leftarrow |x-2|\,|x+2|<\varepsilon$$

$$\Leftarrow |x-2|<\varepsilon/5 \wedge |x+2|<5$$

$$\Leftarrow |x-2|<\varepsilon/5 \wedge |x-2|<1$$

$$\Leftarrow |x-2|<\min\{1, \varepsilon/5\}.$$

Therefore $\delta = \min\{1, \varepsilon/5\}$ is a suitable choice. ■

<u>Ex.</u> Prove $\lim_{x\to 2} \sqrt{x} = \sqrt{2}$.

<u>Proof.</u> We have to prove $\forall \varepsilon>0, \exists \delta>0, 0<|x-2|<\delta \Rightarrow |\sqrt{x}-\sqrt{2}|<\varepsilon$.

Again this means for a given $\varepsilon>0$, we should find a suitable $\delta>0$ such that the above implication holds. Again we try to use a backward argument.

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$$|\sqrt{x}-\sqrt{2}|<\varepsilon \Leftarrow \frac{|x-2|}{|\sqrt{x}+\sqrt{2}|}<\varepsilon.$$

So this time we need an upper bound for the factor $\frac{1}{|\sqrt{x}+\sqrt{2}|}$.

The idea is that when x is fairly close to 2, we expect that $\sqrt{x}+\sqrt{2}$ is fairly close to $2\sqrt{2}$. Hence we should be able to get a lower bound for $|\sqrt{x}+\sqrt{2}|$.

Let's again start we an initial estimate and assume $\delta \le 1$.

Hence $|x-2|<1 \Rightarrow 1<x<3$

$$\Rightarrow 1<\sqrt{x}<\sqrt{3}$$

$$\Rightarrow 1+\sqrt{2}<\sqrt{x}+\sqrt{2}<\sqrt{3}+\sqrt{2}$$

$$\Rightarrow 1<|\sqrt{x}+\sqrt{2}| \Rightarrow \frac{1}{|\sqrt{x}+\sqrt{2}|}<1.$$

Therefore $|\sqrt{x}-\sqrt{2}|<\varepsilon \Leftarrow \frac{|x-2|}{|\sqrt{x}+\sqrt{2}|}<\varepsilon$

$$\Leftarrow |x-2|<\varepsilon \wedge \frac{1}{|\sqrt{x}+\sqrt{2}|}<1$$

$$\Leftarrow |x-2|<\varepsilon \wedge |x-2|<1$$

$$\Leftarrow |x-2|<\min\{1,\varepsilon\}.$$

Thus $\delta=\min\{1,\varepsilon\}$ is a suitable choice. ■

Lecture 09: Limit does not exist.

Friday, October 28, 2016 9:25 AM

To say a limit does not exist, one has to use four quantifiers. That might be why students often have a hard time showing why a limit does not exist.

$\lim_{x\to a} f(x)$ does not exist if and only if for every $L\in\mathbb{R}$, $\lim_{x\to a} f(x)\neq L$.

This means $\forall L\in\mathbb{R},\ \neg(\forall\varepsilon>0,\ \exists\delta>0,\ 0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon)$.

$\exists\varepsilon>0,\ \forall\delta>0,\ \neg(\forall x,\ 0<|x-a|<\delta\Rightarrow|f(x)-L|<\varepsilon)$

An implication $P\Rightarrow Q$ fails exactly when the hypothesis P holds and the conclusion fails; that means $\neg(P\Rightarrow Q)\equiv P\wedge(\neg Q)$. Alternatively, $\neg(P\Rightarrow Q)\equiv\neg((\neg P)\vee Q)\equiv P\wedge(\neg Q)$. Altogether, $\lim_{x\to a} f(x)$ does not hold precisely when

$$\forall L\in\mathbb{R},\ \exists\varepsilon>0,\ \forall\delta>0,\ \exists x,\quad 0<|x-a|<\delta\ \wedge\ |f(x)-L|\geq\varepsilon$$

there exists $x\neq a$ which is δ-close to a and $f(x)$ is ε-away from L.

In the next lecture, we will prove that $\lim_{x\to 0}\sin(\frac{1}{x})$ does not exist. You can also think about the next exercise.

<u>Ex.</u> Suppose $f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$. Then, for every $a\in\mathbb{R}$, $\lim_{x\to a} f(x)$ does not exist.