Lecture 11: Identity function

Wednesday, November 2, 2016 9:19 AM

For every non-empty set X, the identity function I_X of X is

$I_X : X \to X$, $I_X(x) = x$ for every $x \in X$.

<u>Lemma</u> For every function $f : X \to Y$, we have

$$I_Y \circ f = f = f \circ I_X .$$

<u>Proof</u>.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "I_Y \circ f"'] & Y \arrow[d, "I_Y"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[dr, mapsto] & f(x) \arrow[d, mapsto] \\ & f(x) \end{tikzcd}$$

$f : X \to Y$ $\quad x \mapsto f(x)$

$I_Y \circ f : X \to Y$

$x \mapsto I_Y(f(x)) = f(x)$.

So they have the same (co)domain and rules. Hence they are equal functions.

$$\begin{tikzcd} X \arrow[r] & Y \\ X \arrow[u, "I_X"] \arrow[ur, "f \circ I_X"'] & \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] & f(x) \\ x \arrow[u, mapsto] \arrow[ur, mapsto] & \end{tikzcd}$$

$(f \circ I_X)(x) = f(I_X(x))$

$= f(x)$.

And so again $f \circ I_X$ and f have the same (co)domain and rules. Thus $f \circ I_X = f$. ■

<u>Remark</u>. We say this diagram commutes: it does NOT matter which path we choose.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Lecture 11: Example of composite functions

Thursday, November 3, 2016 1:14 AM

Ex. Complete the missing information, if any.

$$f_1(x) = x+1, \quad f_2(x) = \sqrt{x}, \quad \text{and}$$

$$f_2 \circ f_1 : \mathbb{R}^{\geq 0} \longrightarrow \mathbb{R}, \quad f_2 \circ f_1(x) = \sqrt{x+1}.$$

Solution. Domains and co-domains of f_1 and f_2 are missing.

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[dr, "f_2 \circ f_1"'] & Y \arrow[d, "f_2"] \\ & Z \end{tikzcd}$$

Since $f_2 \circ f_1$ exists, codomain of f_1 is the same as domain of f_2. Let's denote it by Y.

domain of f_1 = domain of $f_2 \circ f_1 = \mathbb{R}^{\geq 0}$.

codomain of f_2 = codomain of $f_2 \circ f_1 = \mathbb{R}$.

- So $f_1 : \mathbb{R}^{\geq 0} \longrightarrow Y$, $f_1(x) = x+1$. In particular, $\forall x \in \mathbb{R}^{\geq 0}$, $f_1(x) = x+1 \in Y$. So

$$x \in \mathbb{R}^{\geq 1} \Rightarrow x \in Y \quad \text{and so} \quad \mathbb{R}^{\geq 1} \subseteq Y.$$

- $f_2 : Y \longrightarrow \mathbb{R}$, $f_2(y) = \sqrt{y}$ in order to get a function which is defined at every $y \in Y$, we should assume

$$y \in Y \Rightarrow y \geq 0 \Rightarrow y \in \mathbb{R}^{\geq 0}. \quad \text{Thus} \quad Y \subseteq \mathbb{R}^{\geq 0}.$$

Hence Y can be any set $\mathbb{R}^{\geq 1} \subseteq Y \subseteq \mathbb{R}^{\geq 0}$. $\blacksquare$

Lecture 11: Image of a function

Thursday, November 3, 2016 8:27 AM

<u>Definition</u>. Let $X \xrightarrow{f} Y$. Image of f is a subset of codomain:

$$\mathrm{Im}(f) = \{f(x) \mid x \in X\}.$$

So we have $\boxed{\forall y \in Y,\ (y \in \mathrm{Im}(f) \Leftrightarrow \exists x \in X,\ y = f(x)).}$

<u>Definition</u>. A function $X \xrightarrow{f} Y$ is called <u>surjective</u> or <u>onto</u> if $\mathrm{Im}(f) = Y$.

So we have

$$\boxed{X \xrightarrow{f} Y \text{ is surjective } \Leftrightarrow \forall y \in Y, \exists x \in X, y = f(x)}$$

In another words, for every $y \in Y$, you can <u>solve the</u> equation $y = f(x)$ for $x \in X$.

<u>Ex</u>. Let $f: \mathbb{R}^{\geq 2} \to \mathbb{R}$, $f(x) = x^3$. Find $\mathrm{Im}(f)$.

(we will use the facts that $x \mapsto x^3$ and $x \mapsto \sqrt[3]{x}$ are increasing functions.)

<u>Solution</u>. We claim $\mathrm{Im}(f) = \mathbb{R}^{\geq 8}$. We need to show $\mathrm{Im}(f) \subseteq \mathbb{R}^{\geq 8}$ and $\mathbb{R}^{\geq 8} \subseteq \mathrm{Im}(f)$.

$\mathrm{Im}(f) \subseteq \mathbb{R}^{\geq 8}$. To show this we have to verify

Lecture 11: Image of a function

Thursday, November 3, 2016 8:42 AM

$y \in \mathrm{Im}(f) \overset{?}{\Longrightarrow} y \in \mathbb{R}^{\geq 8}$.

$y \in \mathrm{Im}(f) \Longrightarrow \exists x \in \mathbb{R}^{\geq 2},\ y = f(x) = x^3$.

Since $x \mapsto x^3$ is increasing and $x \geq 2$, we have

$x^3 \geq 8$. So $y \geq 8$, which means $y \in \mathbb{R}^{\geq 8}$.

$\mathbb{R}^{\geq 8} \subseteq \mathrm{Im}(f)$. We have to show

$$y \in \mathbb{R}^{\geq 8} \Rightarrow y \in \mathrm{Im}(f),$$

which means $\forall y \in \mathbb{R}^{\geq 8}, \exists x \in \mathbb{R}^{\geq 2},\ y = x^3$.

If $y \geq 8$, then $x = \sqrt[3]{y} \geq \sqrt[3]{8} = 2$ and $y = x^3$

So for any $y \in \mathbb{R}^{\geq 8}$, $y = (\sqrt[3]{y})^3$ and $\sqrt[3]{y} \in \mathbb{R}^{\geq 2}$. ■

Ex. Suppose A is a non-empty subset of $\mathbb{R}$. Is there a function $f: \mathbb{R} \to \mathbb{R}$ such that $\mathrm{Im}(f) = A$?

Answer. Yes. Since $A \neq \emptyset$, there is $a_0 \in A$. Let

$$f: \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} x & \text{if } x \in A, \\ a_0 & \text{if } x \notin A. \end{cases}$$

Claim. $\mathrm{Im}(f) = A$.

Proof of Claim. As always to show equality of two sets, we have to

Lecture 11: Examples of functions

Wednesday, August 24, 2022 12:19 PM

that one is a subset of the other and vice versa.

$y \in \mathrm{Im}(f) \Rightarrow y = f(x)$ for some $x \in \mathbb{R}$.

<u>Case 1.</u> $x \in A$. In this case, $f(x) = x$; and so $y = x \in A$.

<u>Case 2.</u> $x \notin A$. In this case, $f(x) = a_0$; and so $y = a_0 \in A$.

In either case, we obtain that $y \in A$. Hence $\mathrm{Im}(f) \subseteq A$.

. Next we show that $A \subseteq \mathrm{Im}(f)$.

$x \in A \Rightarrow f(x) = x$ which implies that $x \in \mathrm{Im}(f)$.

Hence $A \subseteq \mathrm{Im}(f)$. Altogether we have $A = \mathrm{Im}(f)$. ∎

In calculus, we often use the graph of f in order to visualize properties of f. We can use the same principle for all functions.

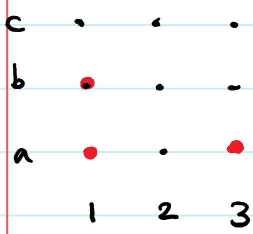
<u>Definition</u>. Graph of $X \xrightarrow{f} Y$ is a subset of $X \times Y$:

$$G_f = \{(x, f(x)) \mid x \in X\}.$$

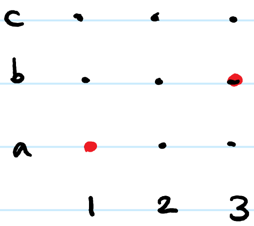
Lecture 11: Examples on image and graph of functions

Friday, November 4, 2016 9:21 AM

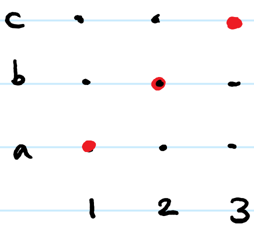
Ex. Which one of the following diagrams represent graph of a function? In each case say whether function is surjective or not?



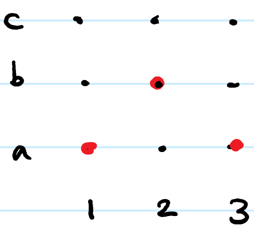
No, it does NOT assign a unique element to 1



No, it does NOT assign any element to 2



Yes, and it is surjective



Yes, but it is NOT surjective. c is NOT in the image

- In graph of a function every "vertical line" intersects the graph in one and exactly one point.

Ex. Suppose $G_f = \{(1,1),(2,3),(4,1)\}$ is graph of a surjective function. Find its domain and codomain.

Solution. First components give us the domain of f and the 2^{nd} components give us the image of f. Since f is surjective we have that codomain $= \mathrm{Im}(f)$. So

domain $= \{1,2,4\}$ and codomain $= \{1,3\}$. ■

Lecture 11: Examples of graph; injective functions

Friday, November 4, 2016 9:34 AM

Definition. A function $f: X \to Y$ is called injective or one-to-one or 1-1 if

$$\forall x_1, x_2 \in X, \left(f(x_1) = f(x_2) \Rightarrow x_1 = x_2\right).$$

Definition. A function $f: X \to Y$ is called bijective if it is both injective and bijective.

- We can use the graph G_f of f to see if it is injective, surjective, or bijective.

f is injective $\Longleftrightarrow$ every horizontal line intersects the graph in at most one point.

f is surjective $\Longleftrightarrow$ every horizontal line intersects the graph in at least one point.

f is bijective $\Longleftrightarrow$ every horizontal line intersects the graph in exactly one point.

Ex. In each case determine whether the given function is injective, surjective, or bijective.

Lecture 11: Injective, bijective functions

Sunday, November 6, 2016 4:34 PM

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x+1$.

(b) $f : \mathbb{R}^{>0} \to \mathbb{R}$, $f(x) = x^2$.

(c) $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, $f(x) = \tan(x)$.

<u>Solution</u>. (a) It is a bijection.

<u>Why is it injective?</u> $\forall x_1, x_2 \in \mathbb{R}, \left(f(x_1) = f(x_2) \overset{?}{\Rightarrow} x_1 = x_2\right)$

$f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$.

<u>Why is it surjective?</u> We have to show

$$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, y = f(x).$$

So for every $y \in \mathbb{R}$, we have to find $x \in \mathbb{R}$ such that $y = x+1$.

We notice $y = x+1 \Leftarrow x = y-1$ and $y-1 \in \mathbb{R}$, which gives us the above claim.

(b) It is injective, but not surjective.

<u>Why is it injective?</u> $\forall x_1, x_2 \in \mathbb{R}^+, \left(f(x_1) = f(x_2) \overset{?}{\Rightarrow} x_1 = x_2\right)$

$f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow |x_1| = |x_2|$

Since $x_1, x_2 \in \mathbb{R}^+$, we have $x_1 = |x_1|$ and $x_2 = |x_2|$ $\Big\} \Rightarrow x_1 = x_2$

<u>Why is it not surjective?</u> For every $x \in \mathbb{R}^+$, $x^2 > 0$. So there is no

Lecture 11: Injective, surjective, bijective

Sunday, November 6, 2016 4:56 PM

$x \in \mathbb{R}^{>0}$ such that $-1 = f(x)$, which implies $-1 \notin \text{Im}(f)$.

Therefore $\text{Im}(f) \neq$ the codomain of f which is $\mathbb{R}$.

[What is $\text{Im}(f)$? Claim: $\text{Im}(f) = \mathbb{R}^+$.

To show this claim, we need to show $\text{Im}(f) \subseteq \mathbb{R}^+$ and

$\mathbb{R}^+ \subseteq \text{Im}(f)$.

Why is $\text{Im}(f) \subseteq \mathbb{R}^+$? We have to show $y \in \text{Im}(f) \Rightarrow y \in \mathbb{R}^+$.

$$y \in \text{Im}(f) \Rightarrow \exists x \in \mathbb{R}^+, y = f(x) \Rightarrow \exists x > 0, y = x^2$$

$$\Rightarrow y > 0 \Rightarrow y \in \mathbb{R}^+.$$

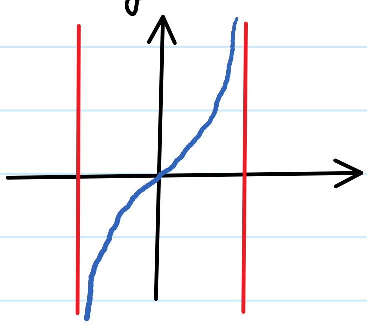
Why is $\mathbb{R}^+ \subseteq \text{Im}(f)$? We have to show $y \in \mathbb{R}^+ \Rightarrow y \in \text{Im}(f)$.

(Backward argument) $y \in \text{Im}(f) \Leftarrow \exists x \in \mathbb{R}^+, y = f(x)$

$\Leftarrow \exists x > 0, y = x^2 \Leftarrow (\sqrt{y} > 0$ and $(\sqrt{y})^2 = y) \Leftarrow y > 0.$]

(c) It is a bijection.

In class, I used graph of $\tan x$ to convey the idea of a proof: As you can see, any horizontal line intersects the graph in one and exactly one point.



Lecture 11: Injection, surjection, bijection

Sunday, November 6, 2016 9:55 PM

Here is a more formal proof using theorems from calculus:

- Function $f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, $f(x) = \tan x$ is a differentiable function and

$$f'(x) = \frac{1}{\cos^2 x} \geq 1 \quad \text{for any } x \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$

(Using mean value theorem, if $x_1 < x_2$, then

$\exists y_0$, $x_1 < y_0 < x_2$ and $\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(y_0) \geq 1$.

In particular, $f(x_2) - f(x_1) \geq x_2 - x_1$). So

if $x_2 > x_1$, then $f(x_2) > f(x_1)$. Therefore f is injective.

We also know $\lim_{x \to \pi/2^+} \tan x = +\infty$ and

$\lim_{x \to \pi/2^-} \tan x = -\infty$. Since tan is continuous, by intermediate value theorem we have $\mathrm{Im}(\tan) = \mathbb{R}$.

(Recall. If $f: [a,b] \to \mathbb{R}$ is continuous, $f(a) < f(b)$, and $f(a) \leq y_0 \leq f(b)$, then there exists $a \leq x_0 \leq b$ such that $f(x_0) = y_0$. (This is called Intermediate value theorem).

(This page is not important for the purposes of this course.)

Lecture 11: Injection, surjection, composition

Monday, November 7, 2016 9:06 AM

Theorem. Suppose $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are two functions.

(a) If $g\circ f$ is injective, then f is injective.

(b) If $g\circ f$ is surjective, then g is surjective.

(c) If f and g are injective, then $g\circ f$ is injective.

(d) If f and g are surjective, then $g\circ f$ is surjective.

Proof. (a) We have to show $\forall x_1, x_2 \in X,\ f(x_1)=f(x_2) \overset{?}{\Longrightarrow} x_1=x_2$.

$$f(x_1)=f(x_2) \Longrightarrow g(f(x_1)) = g(f(x_2))$$
$$\Longrightarrow (g\circ f)(x_1) = (g\circ f)(x_2)$$
$$\Longrightarrow x_1=x_2 \quad \text{since } g\circ f \text{ is injective.}$$

(b) We have to show $\forall z\in Z, \exists y \in Y,\ g(y)=z$. $\circledast$

We know $g\circ f$ is surjective. So

$$\forall z\in Z,\ \exists x\in X,\ (g\circ f)(x)=z, \text{ which implies}$$
$$\forall z\in Z,\ \exists x\in X,\ g(f(x))=z.$$

For a given $z\in Z$, let $x\in X$ be such that $g(f(x))=z$

then $y=f(x)\in Y$ and $g(y)=z$ which implies $\circledast$.

Lecture 11: Injection, surjection, composition

Monday, November 7, 2016 9:21 AM

(c) We have to show $\forall x_1, x_2 \in X$, $(g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow x_1 = x_2$.

$$(g\circ f)(x_1) = (g \circ f)(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$$

$$\Rightarrow f(x_1) = f(x_2) \quad \text{since } g \text{ is injective}$$

$$\Rightarrow x_1 = x_2 \quad \text{since } f \text{ is injective.}$$

(d) We have to show $\forall z \in Z$, $\exists x \in X$, $(g \circ f)(x) = z$, which means $g(f(x)) = z$ for some $x \in X$.

We go "one step at a time":

Since g is surjective, for some $y \in Y$ we have $g(y) = z$.
Choose such y and call it y_0.

Since f is surjective, for some $x \in X$ we have $f(x) = y_0$.
Choose such x and call it x_0.

So we have $g(y_0) = z$ and $f(x_0) = y_0$. Therefore

$(g \circ f)(x_0) = g(f(x_0)) = g(y_0) = z$, as we wished. ■

<u>Corollary</u>. Suppose $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are two functions.
If $g \circ f$ is a bijection, then f is injective and g is surjective.

Lecture 11: bijection and composition

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<u>Corollary</u>. Suppose $f: X \to Y$ and $g: Y \to Z$ are two functions. Then

$$f, g: \text{bijective} \Rightarrow g \circ f: \text{bijective}.$$

<u>Proof</u>.

$$f, g: \text{bijective} \Rightarrow \left\{ \begin{array}{l} f, g: \text{injective} \Rightarrow g \circ f: \text{injective} \\ \\ f, g: \text{surjective} \Rightarrow g \circ f: \text{surjective} \end{array} \right\} \Rightarrow g \circ f: \text{bijective}.$$

$\blacksquare$