Lecture 12: Left and right invertible functions

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Suppose $f: X \to Y$ is a function. We say $g: Y \to X$ is a **right inverse** of f if $f \circ g = I_Y$.

We say $h: Y \to X$ is a **left inverse** of f if $g \circ f = I_X$.

<u>Ex.</u> Let $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 2y$. Then

$g: \mathbb{R} \to \mathbb{R}^2$, $g(y) = (3y, y/2)$ is a right inverse as

$f \circ g: \mathbb{R} \to \mathbb{R}$, $(f \circ g)(y) = f(3y, y/2) = y$. Notice that g is NOT a left inverse as $(g \circ f)(x,y) = (6y, y)$. In fact, f doe not have a left inverse.

We also observe that f is a left inverse of g, and g does not have a right inverse.

<u>Theorem.</u> Suppose $f: X \to Y$ is a function. Then the following statements hold.

(1) f has a right inverse $\Leftrightarrow$ f is surjective.

(2) f has a left inverse $\Leftrightarrow$ f is injective.

<u>Proof</u>. (1) ($\Rightarrow$) Since f has a right inverse, there exists a function

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$h: Y \to X$, $f \circ h = I_Y$. Since I_Y is surjective, f is surjective.

(In the previous lecture we have proved that $f_1 \circ f_2$ is surjective implies that f_1 is surjective.)

($\Leftarrow$) In the proof we will be using an axiom of set theory called <u>axiom of choice</u>. First proof will be written and then it will be mentioned where axiom of chioce is used.

We assume f is surjective. And we have to find $h: Y \to X$ such that $(f \circ h)(y) = y$. So h should be defined in a way such that $f(h(y)) = y$.

For every $y \in Y$, let $\overleftarrow{f}(y) = \{x \in X \mid f(x) = y\}$ be the preimage of y. Since f is surjective, $\overleftarrow{f}(y) \neq \emptyset$ for every $y \in Y$.

Let's <u>choose one element of $\overleftarrow{f}(y)$</u> and call it $h(y)$. So we get a function $h: Y \to X$ such that $h(y) \in \overleftarrow{f}(y)$.

So $\forall y \in Y$, $f(h(y)) = y$. Hence $f \circ h = I_Y$ as both of these functions are from Y to Y and $(f \circ h)(y) = f(h(y)) = y = I_Y(y)$.

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The above argument is almost complete.

When a single set Z is non-empty, we can get $z \in Z$. But to do it simultaneously for a family of non-empty sets, one needs axiom of choice:

Suppose $F: Y \to P(X)$ be a function such that

$\forall y \in Y$, $F(y) \neq \emptyset$.

Then there is a function $h: Y \to X$ such that

$\forall y \in Y$, $h(y) \in F(y)$.

Using the axiom of choice for $F: Y \to P(X)$, $F(y) = \overleftarrow{f}(y)$ we get the desired $h: Y \to X$. ∎

Next we characterize having a left inverse.

Theorem. Suppose $f: X \to Y$ is a function. Then f has a left inverse $\iff$ f is injective.

Proof. ($\Rightarrow$) Since f has a left inverse, there exists $g: Y \to X$ such that $g \circ f = I_X$. Because $g \circ f = I_X$ is injective,

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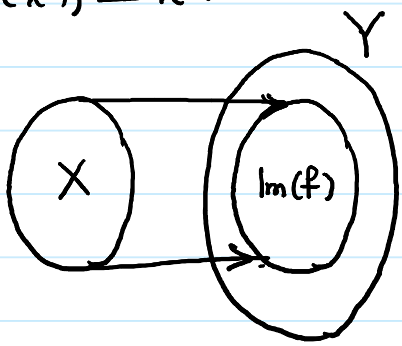
by a theorem that we proved in the previous lecture, f is injective.

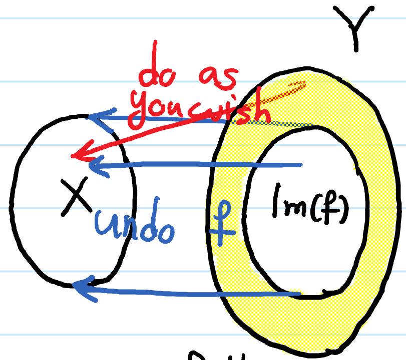
($\Leftarrow$) Suppose $X \xrightarrow{f} Y$ is injective. We would like to define a function $Y \xrightarrow{g} X$ such that $g \circ f = I_X$, which means, for every $x \in X$, $g(f(x)) = x$.

This means g should undo f on the image of f and can be anything outside of $\mathrm{Im}(f)$.



Here is a formal definition:

Choose $x_0 \in X$ (we can do that since $X \neq \emptyset$). Define $Y \xrightarrow{g} X$ as follows:



$$g(y) = \begin{cases} x & \text{if } y = f(x) \text{ for some } x \in X \\ x_0 & \text{if } y \in Y \setminus \mathrm{Im}(f). \end{cases}$$

We need to show g is a function (we say g is well-defined).

And then we have to check that $g \circ f = I_X$.

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[Recall that to show "an assigning rule" defines a function from X to Y, we have to check three things:

1. This "rule" can be applied to all the elements of X.
2. This "rule" assigns elements of Y to every element of X.
3. This "rule" assigns a unique element of Y to every element of X.

For instance, we have seen that $f: \mathbb{R}^+ \to \mathbb{R}$, $f(x) = y$ if $y^2 = x$ does NOT define a function. This rule assigns two elements of $\mathbb{R}$ to 1. Both 1 and -1 are assigned to 1.]

Well-definedness of g It clearly assigns elements of Y to every element of X. We have to check why it assigns a unique element:

- If $y \in Y \setminus \mathrm{Im}(f)$, then x_0 is assigned to y with no ambiguity
- Suppose $y \in \mathrm{Im}(f)$, and x_1 and x_2 can be assigned to y. So $f(x_1) = y \wedge f(x_2) = y$, which implies $f(x_1) = f(x_2)$. Since f is injective and $f(x_1) = f(x_2)$, we obtain that $x_1 = x_2$. So a unique element of X is assigned to y.

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Checking $g\circ f = I_X$.

Both $g\circ f$ and I_X are functions from X to X. So we have to check only that $(g\circ f)(x) = I_X(x)$ for all $x\in X$.

$$\begin{aligned}(g\circ f)(x) &= g(f(x)) = g(y) && \text{where } y=f(x)\\ &= x && \text{the way we defined } g.\\ &= I_X(x). && \blacksquare\end{aligned}$$

Theorem. Suppose $f: X\to Y$ is a function. Then the following statements hold.

(a) If g is a right inverse of f and h is a left inverse of f, then $g=h$.

(b) f is a bijection $\iff \exists g: Y\to X$, $f\circ g = I_Y \wedge g\circ f = I_X$.

Proof. We will prove part (b) in the next lecture. You will see a more general version of part (a) in your algebra courses.

- g: right inverse $\Rightarrow g: Y\to X$ and $f\circ g = I_Y$ (1)
- h: left inverse $\Rightarrow h: Y\to X$ and $h\circ f = I_X$ (2)

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By (1) and (2), we have

$$g = I_X \circ g = (h \circ f) \circ g = h \circ (f \circ g) = h \circ I_Y = h.$$

- We also observe that the previous theorems imply:

f has both left and right inverses $\Leftrightarrow$ f is bijective.

(We will continue in the next lecture.)