Math 109 Homework 1 Solutions

1. We check the truth tables for these statements:

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

P	Q	R	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	F	F	F

Since the final columns match, the two statements are equivalent.

2. Using the disjunctive form of an implication, we know:

$$P \vee Q \Rightarrow R \equiv \neg(P \vee Q) \vee R$$

Applying De Morgan's laws to the $\neg(P \vee Q)$ gives:

$$\neg(P \vee Q) \vee R \equiv (\neg P \wedge Q) \vee R$$

Applying problem 1 (and using $A \vee B \equiv B \vee A$) gives:

$$(\neg P \wedge \neg Q) \vee R \equiv (\neg P \vee R) \wedge (\neg Q \vee R)$$

Lastly, applying the disjunctive form of an implication twice gives:

$$(\neg P \vee R) \wedge (\neg Q \vee R) \equiv (P \Rightarrow R) \wedge (Q \Rightarrow R)$$

and combining this chain of equivalences gives the desired result.

3. To show that $P \Rightarrow Q \not\equiv Q \Rightarrow P$, it suffices to find one truth assignment to P and Q for which the truth values of $P \Rightarrow Q$ and $Q \Rightarrow P$ differs. One possibility is P true and Q false, in which case $P \Rightarrow Q$ is false while $Q \Rightarrow P$ is true.

4. (a) We proceed by cases:

Case 1. $a \geq 0$. Then $|a| = a$, so $|a| \geq a$.

Case 2. $a < 0$. Then $|a| = -a$. Since $a < 0$, adding $-a$ to both sides gives $a + (-a) < 0 + (-a)$, or $0 < -a$. By the transitive property, we have $a < 0 < -a$, so $a < |a|$, and $a \leq |a|$. Since we have shown it is true that $|a| \geq a$ in all possible cases, we are done.

(b) We again proceed by cases:

Case 1. $b \geq 0$. Then $|b| = b$, so $|b|^2 = b^2$.

Case 2. $b < 0$. Then $|b| = (-b)$, so $|b|^2 = (-b)(-b) = -(b(-b)) = -(-(b^2)) = b^2$.

In either case, we find $|b|^2 = b$, so we are done.

(Note that we are using the fact that for all real numbers a, b, $(-a)b = -(ab)$, which is not directly part of our axioms, see page 7 of lecture notes 1 for how to show this)

(c) By a very similar argument to part (a), it is also true that for any real number a, $-a \leq |a|$.

We now proceed by cases:

Case 1. $(c + d) \geq 0$. Then $|c + d| = c + d$. By part (a), $c \leq |c|$, so $c + d \leq |c| + d$. Also by part (a), $d \leq |d|$, so $|c| + d = d + |c| \leq |d| + |c| = |c| + |d|$. By transitivity, $c + d \leq |c| + |d|$, and therefore $|c + d| \leq |c| + |d|$.

(For simplicity's sake, we have assumed that the axioms stated for strict inequalities ($<$) also hold for non-strict inequalities ($\leq$). You should double check you know how you would prove this)

Case 2. $(c+d) < 0$. Then $|c+d| = -(c+d) = (-c)+(-d)$. Since $-c \leq |c|$, we have $(-c)+(-d) \leq |c|+(-d)$, and since $-d \leq |d|$ we have $|c| + (-d) \leq |c| + |d|$, so by transitivity we have $(-c) + (-d) \leq |c| + |d|$, and therefore $|c + d| \leq |c| + |d|$.

(Note that the claim $-(c + d) = (-c) + (-d)$ is again not directly part of our axioms, check that you can see how to prove it)

Since the claim holds in all cases, we are done.

5. We can use the disjunctive form of an implication and De Morgan's laws to check that all three are equivalent to $\neg P \vee Q \vee R$:

$$P \Rightarrow (Q \vee R) \equiv \neg P \vee (Q \vee R) \equiv \neg P \vee Q \vee R$$
$$(P \wedge \neg Q) \Rightarrow R \equiv \neg(P \wedge \neg Q) \vee R \equiv (\neg P \vee \neg\neg Q) \vee R \equiv \neg P \vee Q \vee R$$
$$P \wedge (\neg Q) \wedge (\neg R)) \Rightarrow \perp \equiv \neg(P \wedge (\neg Q) \wedge (\neg R))) \vee \perp \equiv (\neg P \vee Q \vee R) \vee \perp \equiv \neg P \vee Q \vee R$$

6. Suppose d, m_1, m_2, n_1, n_2 are integers such that $d \mid (m_1 - m_2)$ and $d \mid (n_1 - n_2)$. Then there exist integers k_1 and k_2 such that

$$m_1 - m_2 = k_1 d \qquad \text{and} \qquad n_1 - n_2 = k_2 d$$

Then

$$(m_1 + n_1) - (m_2 + n_2) = (m_1 - m_2) + (n_1 - n_2) = k_1 d + k_2 d = (k_1 + k_2)d$$

Since $k_1 + k_2$ is an integer, we have $d \mid (m_1 + n_1) - (m_2 + n_2)$.

Also,

$$
\begin{aligned}
m_1n_1 - m_2n_2 &= (m_1n_1 - m_2n_2) + 0 \\
&= (m_1n_1 - m_2n_2) + (m_1n_2 - m_1n_2) \\
&= m_1n_1 - m_1n_2 + m_1n_2 - m_2n_2 \\
&= m_1(n_1 - n_2) + (m_1 - m_2)n_2 \\
&= m_1k_2d + n_2k_1d \\
&= (m_1k_2 + n_2k_1)d
\end{aligned}
$$

Since $m_1k_2 + n_2k_1$ is an integer, we have $d \mid m_1n_1 - m_2n_2$.