M109 HW2 Solutions

1 A.3

We want to prove that $L_1 = L_2$, equivalently that $L_1 - L_2 = 0$. Denote this value by x; now we want to show $(\forall \varepsilon > 0 : \ |x| < \varepsilon) \implies x = 0$. We prove the implication by contradiction. Suppose that the assumption is true but the conclusion is not; that is, suppose $(\forall \varepsilon > 0 : |x| < \varepsilon) \wedge x \neq 0$. Since $x \neq 0$, $|x| > 0$, so using it as ε we get $|x| < |x|$, which is the desired contradiction.

2 A.4

(a) $\forall \varepsilon > 0, \exists N : \forall n > N : (|x_n - a| < \varepsilon)$.
(b) Assume it does exist and denote it by L. We will show that L is then arbitrarily close to both L_1 and L_2, which will be absurd. By the definition of a sequential limit, $\exists N' : \forall n > N' : |f(x_n^-) - L_1| < \varepsilon/2$. By the definition of the limit of f, we know that for some $d > 0$, the implication $|x - a| < d \implies |f(x) - L| < \varepsilon/2$ holds. Since $f(x_n^-) \to a$, starting from some N'' we have $|x_n^- - a| < d$, and then $|f(x_n^-) - L| < \varepsilon/2$ by the implication just mentioned. Then starting from $N := \max(N', N'')$, $|f(x_n^-) - L| < \varepsilon/2$ and $|f(x_n^-) - L_1| < \varepsilon/2$. Combining these two together leads to $|L - L_1| = |(L - f(x_n^-)) - (L_1 - f(x_n^-))| \leq |(L - f(x_n^-))| + |(L_1 - f(x_n^-))| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for any $n > N$. Then the previous problem implies $L_1 = L$. Similarly, $L_2 = L$. This contradicts $L_1 \neq L_2$.

[Those attending the 9am discussion will notice that the problem immediately follows from the Heine definition of the limit.]

3 B.1

By definition, $A \times (B \cup C) = \{(a, x) | a \in A \wedge (x \in B \vee x \in C)\} = \{(a, x) | (a \in A \wedge x \in B) \vee (a \in A \wedge x \in C)\} = \{y | y \in A \times B \vee y \in A \times C\} = A \times B \cup A \times C$.

4 B.2

(a) See 1.
(b) We disprove it by a counterexample: let $x = 1, \varepsilon = 2$. Then the implication is false: the assumption is true but the conclusion is not.

5 B.3

(a) $x \in A \cap B$ is equivalent to $x \in A \wedge x \in B$, which is equivalent to $\mathbb{1}_A(x) = 1 \wedge \mathbb{1}_B(x) = 1$, which is equivalent to $\mathbb{1}_A(x) \cdot \mathbb{1}_B(x) = 1$.
(b) Any $x \in X$ is *either* in A *or* in A^c, so exactly one of $\mathbb{1}_A$ and $\mathbb{1}_{A^c}$ will assume value 1 on x.
(c) If x is in $A \triangle B$, we get $1 + 0 - 0$ (or $0 + 1 - 0$), just as we want for $\mathbb{1}_{A \cup B}(x)$. If $x \in A \cap B$, we get $1 + 1 - 1 = 1 = \mathbb{1}_{A \cup B}(x)$ Finally, if $x \notin A \wedge x \notin B$, we get $0 + 0 - 0 = \mathbb{1}_{A \cup B}(x)$.
(d) $\mathbb{1}_{A \setminus B} = \mathbb{1}_{A \cap B^c} = \mathbb{1}_A \cdot \mathbb{1}_{B^c} = \mathbb{1}_A \cdot (\mathbb{1}_X - \mathbb{1}_B) = \mathbb{1}_A - \mathbb{1}_A \cdot \mathbb{1}_B$, where in the last equality we distribute and use that $\mathbb{1}_X$ is the function constantly equal to 1.
(e) If $x \in A \triangle B$, we get $1 + 0 - 2 \cdot 0$ (or $0 + 1 - 2 \cdot 0$), which happens to equal $\mathbb{1}_{A \triangle B}(x)$. If $x \in A \cap B$, we get $1 + 1 - 2 = 0$. Finally, if $x \in (A \cup B)^c$, we get $0 + 0 - 0$.
(f) $\forall x, \mathbb{1}_A(x) \leq \mathbb{1}_B(x)$ is equivalent to $\forall x, \mathbb{1}_A(x) = 1 \implies \mathbb{1}_B(x) = 1$, which is equivalent to $\forall x, x \in A \implies x \in B$ which is equivalent to $A \subset B$.

6 B.4

The idea is that a subset $S \subset X$ is *uniquely* encoded by the ability to answer for any $x \in X$ whether or not $x \in S$. This, we prove that θ is a bijection by constructing a two-sided inverse, which would "decode" the subset S back from the piece knowledge described above.

Define $\phi : F(X, \{0,1\}) \to P(X)$ by $\phi(f) := \{x \in X | f(x) = 1\}$. Then $\phi \circ \theta(S) = \{x \in X | \mathbb{1}_S(x) = 1\} = S$, and $(\theta \circ \phi)(f) = \mathbb{1}_{\{x \in X | f(x)=1\}}$. The latter is a function which is 1 on x precisely when $x \in \{x \in X | f(x) = 1\}$ which tautologically amounts to the condition $f(x) = 1$. Thus this function equals f.

We've shown that both $\phi \circ \theta$ and $\theta \circ \phi$ send their inputs to themselves, hence proving that ϕ is a two-sided inverse of θ.

7 C.1

(a) In this sum, nonzero summands correspond bijectively to elements of A. Thus, the sum equals $|A|$.
(b) Immediately follows from summing the functions from both sides of the equality over all $x \in X$, as in (a).
(c) The base case $n = 1$ is $\mathbb{1}_{A^c} = 1 - \mathbb{1}_A$, which was proven above. Assuming this formula holds up to $n - 1$, let's prove it for n. We have $\mathbb{1}_{(A_1 \cup \ldots \cup A_n)^c} = \mathbb{1}_{((A_1 \cup \ldots \cup A_{n-1}) \cup A_n)^c} = \mathbb{1}_{(A_1 \cup \ldots \cup A_{n-1})^c \cap A_n^c} = \mathbb{1}_{(A_1 \cup \ldots \cup A_{n-1})^c} \cdot \mathbb{1}_{A_n^c} = ((1 - \mathbb{1}_{A_1}) \cdot \ldots \cdot (1 - \mathbb{1}_{A_{n-1}}))(1 - \mathbb{1}_{A_n})$, which is what we needed. Here the first equality was inspired by our desire to use the inductive hypothesis, the second one uses the first equality from the hint, the third one uses the third equality from the hint, and the last one uses the inductive hypotheses.
(d) The equality with characteristic functions is obtained by directly expanding the brackets. When we do this, for each summand we take 1's from some of the multiples $(1 - \mathbb{1}_{A_i})$, and $-\mathbb{1}_{A_i}$'s from the rest. Thus we will get all possible products $\mathbb{1}_{A_{i_1}} \cdot \ldots \cdot \mathbb{1}_{A_{i_k}}$, each multiplied by (-1) k times, just as the formula says.
Now, summing these equal functions over all $x \in X$, just like we did in (b), gives us the inclusion-exclusion formula.

8 C.2

(a) $g = g \circ \mathrm{id}_X = g \circ (f \circ h) = (g \circ f) \circ h = \mathrm{id}_X \circ h = h$.
(b) By the theorem from the lecture[1], f, being an injection and a surjection, correspondingly has left and right inverses. Now, by (a) they have to coincide.

[1]the first page of https://mathweb.ucsd.edu/~asalehig/math109-ss-22-lecture12.pdf