1. (Double cosets) Suppose $G$ is a group, and $H, K \leq G$. For any $g \in G$, let $HgK = \{ hgk \mid h \in H, k \in K \}$. Notice that $H \backslash G / K$ by left translations, and $H \backslash G \rhd K$ by right translations. Convince yourselves that

$$HgK = \text{union of elements of the } H\text{-orbit of } gK$$
and

$$HgK = \text{union of elements of the } K\text{-orbit of } Hg.$$ 

(a) Show that $\{ HgK \mid g \in G \}$ is a partition of $G$.

This partition is denoted by $H \backslash G / K$.

(b) Show that there are bijections between the quotient spaces $H \backslash (G / K)$, $(H \backslash G) / K$, and $H \backslash G / K$.

(c) Show that $H / H \cap gKg^{-1} \rightarrow HgK / K$,

$$h(H \cap gKg^{-1}) \mapsto hgK$$

is a bijection; in particular, if $|G| < \infty$, then

$$|HgK| = |K| |H| / |H \cap gKg^{-1}|.$$ 

(d) Let $G = \text{SL}_2(\mathbb{Z}/p\mathbb{Z})$ and $B = \{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \mid a \in (\mathbb{Z}/p\mathbb{Z})^*, b \in \mathbb{Z}/p\mathbb{Z} \}$. Find $|B \backslash G / B|$.
(Hint: For part (1) notice that $B \sim$ projective space $\mathbb{P}(F^2)$ has two orbits: $\mathbb{E} \{ (0,0) \} \text{ and } \mathbb{E} \{ (a,1) \} \text{ if } a \in F \neq 0$. Let $\mathbb{E}$ any field.

And $S_3(F)/B$ is in bijection with the projective space $\mathbb{P}(F^2)$.

Here $\mathbb{P}(F^2) = \mathbb{E} \{ (a,b) \} \mid (a,b) \in \mathbb{E} \{ (0,0) \} \text{ and }$ where $\mathbb{E} \{ (a,b) \}$ is the line which passes through $(0,0)$ and $(a,b)$.

2. Suppose $G$ is a group, and $|G| = p(p+1)$ where $p$ is an odd prime. Suppose $G$ has more than 1 Sylow $p$-subgroup. Prove that $p$ is a Mersenne prime; that means $p = 2^n - 1$ for some positive integer $n$.

(Hint: Go over the proof presented in class, use Cauchy’s theorem, and the fact that $2 \mid p+1$.)

3. Suppose $G$ is a finite group, and $N \triangleleft G$. Let $P \in \text{Syl}_p(G)$.

Prove that $G = N \cdot G(P) \cdot N$.

(Hint: Show that $G \cap \text{Syl}_p(N)$ by conjugation, and then use Sylow’s 2nd theorem.)
4. Suppose $G \leq X$ transitively. Prove that the kernel of this group action is the normal core $\text{core}(G_x)$ of the stabilizer group of a point $x \in X$.

5. Suppose $G$ is a group of order $pq\ell$ where $p$, $q$, and $\ell$ are distinct prime numbers. Prove that $G$ has a normal subgroup of prime order.

(Hint. Using the contrary assumption, show

$|\text{Syl}_p(G)| = p$, $|\text{Syl}_q(G)| \geq p$, $|\text{Syl}_\ell(G)| \geq q$.

And get a lower bound larger than $|G|$ for

$|\{g \in G \mid \text{o}(g) \text{ is either } p, \text{ or } q, \text{ or } \ell \}|$.)

6. Suppose $G$ is a finite $p$-group and $g \in G$. Prove that $N \cap Z(G) \neq \{g\}$.

7. Suppose $G$ is a finite group, $H \triangleleft G$, and $p$ is a prime factor of $|H|$. Suppose $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_p(H)$. Prove that

$\exists g \in G \text{ st. } Q = gPg^{-1} \cap H$. 

\textbf{6.} Prove that the following is a well-defined surjective function

\[ \phi : \text{Syl}_p(G) \rightarrow \text{Syl}_p(H), \]

\[ \Phi \mapsto \Phi \cap H. \]

\textbf{7.} For \( \Phi \in \text{Syl}_p(G) \), show that \( G = N_G(\Phi H) H \), and conclude

\[ |\text{Syl}_p(G)| = \frac{|N_G(\Phi H)|}{|N_G(\Phi)|} \cdot \frac{|H|}{|N_H(\Phi H)|}. \]

\textbf{8.} Prove that \( |\text{Syl}_p(H)| \mid |\text{Syl}_p(G)| \).