Groups are symmetries of objects. Let’s see a few examples to understand this sentence better.

At the level of set theory.

Let \( X \) be a set. As \( X \) does not have a particular structure any bijection \( f: X \to X \) is a symmetry! This group is denoted by \( S_X \), and is called the symmetric group of \( X \).

\[
S_X := \{ f: X \to X \mid f \text{ is a bijection} \}.
\]

For a positive integer \( n \), we write \( S_n \) instead of \( S_{\{1, 2, \ldots, n\}} \). You have seen before that

\[
|S_n| = n! = 1 \times 2 \times \cdots \times n.
\]

Euclidean plane.

Let \( E \) be the Euclidean plane; this means as a set \( E = \mathbb{R}^2 \); but it also has the Euclidean distance. Symmetries of \( E \) are \( \{ T: E \to E \mid T \text{ bijective} ; \text{ preserves distance} \} \).
Euclid characterized all the elements of $\text{Symm. of } E$.
He showed any symmetry can be achieved as a combination of a translation, a rotation, and/or a reflection about a line.

Q: What is the order of a reflection?
A: 2

Q: What is the order of a translation?
A: Infinity (it is NOT a torsion element.)

Q: What is the order of a rotation of angle $\alpha$?
A: It depends on $\alpha$.
A rotation of angle $\alpha$ is torsion; that means it has a finite order $\iff \frac{\alpha}{2\pi}$ is a rational number (why?)

Using linear algebra, one can write Euclid’s result as

$\text{Symm. of the Euclidean plane} = \{ T: \mathbb{R}^2 \to \mathbb{R}^2 | T_{v} = K_{v} + b \text{ where} \}

K \text{ is orthogonal; } K^T K = I
b \in \mathbb{R}^2 \}$.

Exercise: Prove that any symm. of the Euclidean plane is a
Combination of a translation, a rotation, and/or a reflection.

**Hint.** The key property is the following rigidity of the Euclidean plane:

Suppose $A, B, C$ are three points that are not collinear. Then any point $D$ is uniquely determined by its distance from $A$, $B$, and $C$.

$$D \mapsto (|AD|, |BD|, |CD|)$$

is a bijection.

CGPS works because of a similar reason.

This rigidity implies that, if a symmetry $\phi$ of the Euclidean plane fixes $(0,0)$, $(1,0)$, and $(0,1)$, then $\phi$ is the identity map.

Now for an arbitrary symmetry $\phi : E \to E$,

first we compose $\phi$ with a translation to make sure that $(0,0)$ is fixed; second compose it with a rotation about $(0,0)$ to make sure $(1,0)$ is fixed, too.
Now that \((0,0)\) and \((1,0)\) are fixed, \((0,1)\) is either sent to itself or to \((0,-1)\).

Hence by composing with a reflection, if needed, we can get that

\[
L \cdot R \cdot T \cdot \Phi \text{ fixes the triangle } (0,0), (1,0), (0,1). \text{ Therefore it is the identity map.}
\]

**Symmetries of a graph.**

Let \(G = (V, E)\) be a graph. Then the group of symmetries of \(G\) is denoted by \(\text{Aut}(G)\):

\[
\text{Aut}(G) = \{ f : V \to V \mid f \text{ is a bijection}; \forall v, w, v, w \in V, \exists e, w, g \in E \iff \exists e, f(v), f(w) \in E \}
\]

\(v\) is connected to \(w\)

\(f(v)\) is connected to \(f(w)\).

In many instances, we would like to show that the group of
symmetries of an object determines the object in a unique way. This is how Klein wanted to classify "geometries"; and this point of view is crucial in Galois theory.

**Example** Give some elements of Symm(5)

\[
\begin{array}{c}
\text{rotation. } \text{So } \tau^5 = \text{id.} \\
\text{(No fixed point on the graph.)}
\end{array}
\]

\[
\begin{array}{c}
\text{reflection. } \text{So } \sigma^2 = \text{id.} \\
\text{(Has exactly one fixed point in the set of vertices)}
\end{array}
\]

**Q** Do \( \sigma \) and \( \tau \) commute?

**A** To answer this question we have to look at \( \tau \sigma \tau^{-1} \) and find out if it is \( \sigma \) or not. (\( \tau \sigma \tau^{-1} \) is called a conjugate of \( \sigma \); we have conjugated \( \sigma \) by \( \tau \).)

A good technique is looking at the fixed point of \( \sigma \):
We know $\sigma(1) = 1$ and $\tau(1) = 2$. So

$$\sigma((\tau^{-1}(2))) = 1,$$

which implies

$$(\tau \circ \sigma \circ \tau^{-1})(2) = 2.$$ 

So the fixed point of $\tau \circ \sigma \circ \tau^{-1}$ is different from $\sigma$, which implies $\tau \circ \sigma \circ \tau^{-1}$. Looking at the graph, we can see that $\tau \circ \tau^{-1}$ can be described $\tau \circ \tau^{-1}$ as the following reflection:

One can see that if a symmetry of does not have a fixed point it is a rotation; and if it is not identity and it fixes a point, then it is a "reflection". So $|\text{Symm}(\bigcirc)\bigcirc| = 10$.

**Def.** $\text{Symm}(\bigcirc)$ is called the dihedral group $D_{2n}$.

**Exercise.** Show that $|D_{2n}| = 2n$; $n$: rotations and $n$: reflections.
So far we have started with an object $X$, and then considered
the group of symm. of $X = \{ f : X \rightarrow X \mid f \text{ is a bijection and } f \text{ preserves the structure of } X \}$.

Next we would like to make this abstract:

**Def.** Let $G$ be a group and $X$ be a set. A (left) action
of $G$ on $X$ is $m : G \times X \rightarrow X$, $m(g, x) = g \cdot x$
which has the following properties:

1. $e \cdot x = x$ for any $x \in X$ where $e$ is the
   neutral element of $G$.

2. $\forall x \in X, \forall g_1, g_2 \in G$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.

we say $G$ acts on $X$, and write $G \curvearrowright X$.

**Important example.**

Let $X$ be any object; think about just a set, Euclidean
plane, a graph, etc. Then $\operatorname{Symm}(X) \curvearrowright X$.

**Proof.** $f \in \operatorname{Symm}(X) \iff f : X \rightarrow X$ is a bijection and
$f$ preserves the structure of $X$. 
Now we need to define the group action map

\[ \text{Symm}(X) \times X \rightarrow X \]

\[ (f, x) \mapsto ? \]

The group action should tell us what the group element \( f \)
does to the point \( x \).

As soon as we phrase the question in this way, we
would be forced to think about \( f(x) \) as a possible
answer. And it is:

Let \( m: \text{Symm}(X) \times X \rightarrow X \), \( m(f, x) = f(x) \).

Then \( m(I_X, x) = I_X(x) = x \).

The identity function of
\( X \) is the neutral element
of \( \text{Symm}(X) \)

\[ \forall f_1, f_2 \in \text{Symm}(X), \quad m(f_1, m(f_2, x)) = m(f_1, f_2(x)) = f_1(f_2(x)) \]

\[ = (f_1 \circ f_2)(x). \]

\[ = m(f_1 \circ f_2, x). \]

Ex. \( S_n \subset \{1, 2, ..., n\} \); \( \text{GL}_n(\mathbb{R}) \subset \mathbb{R}^{n \times n} \), \( (A, x) \mapsto A \cdot x \)
The following is an important point of view towards functions \( G \times X \to X \) (here we are not assuming any special property for \( G, X, \) or \( m \)).

For any such function, we can fix the first component \( g \) and get a function \( m_g : X \to X \). This way we get a family \( \{ m_g \}_{g \in G} \) of functions \( m_g : X \to X \).

And this can be reversed:

\[
\begin{array}{ccc}
G \times X & \xrightarrow{m} & X \\
\downarrow & & \downarrow \\
\{ m_g \}_{g \in G} & \to & X \\
\end{array}
\]

where \( m_g : X \to X \)

\[
m_g(x) = m(x)
\]

is a bijection.

Now we would like to know what happens if \( m : G \times X \to X \) is a group action. In the next lecture we will prove

**Theorem.** There is a bijection between

\[
\{ \text{group action } m : G \times X \to X \} \quad \text{and} \quad \text{Hom}(G, S_X).
\]

(In fact, the function given in \( \oplus \) induces a bijection.)