Def. Suppose $G$ is a finite group, $p^n | |G|$ and $p^{n+1} \nmid |G|$. Then a subgroup $P$ of order $p^n$ is called a Sylow $p$-subgp of $G$. And $\text{Syl}_p(G) = \{P \leq G \mid P \text{ is a Sylow } p\text{-subgp}\}$.

So the 1st Sylow theorem implies $\text{Syl}_p(G) \neq \emptyset$.

Observe that $G \cong \text{Syl}_p(G)$ by conjugation.

**Theorem.** $G \cong \text{Syl}_p(G)$ is a transitive action; that means any two Sylow $p$-subgroups are conjugate.

We prove the following stronger version:

(Sylow’s 2nd thm) **Theorem** Suppose $P'$ is a $p$-subgp of $G$, and $P \in \text{Syl}_p(G)$.

Then $\exists g \in G$, $P' \leq gPg^{-1}$.

$P' \cong G/P$ by the left translations. Since $P'$ is a $p$-gp,

$$|G/P| = |(G/P)^{P'}| \mod p.$$ 

$gP \in (G/P)^{P'} \iff \forall P' \in P', \; P'gP = gP$

$\iff \forall P' \in P', \; g^{-1}P'g \in P$

$\iff g^{-1}Pg \leq P \iff P \leq gPg^{-1}$. 


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So \( P' \leq \text{a conjugate of } P \iff (G/P)^P \neq \emptyset \).

Since \( p \nmid |G/P| \), by \( \varpi \) \( p \mid |(G/P)^P| \). And so \( (G/P)^P \neq \emptyset \). \( \Box \)

Corollary. Suppose \( P \in \text{Syl}_p(G) \). Then \( \text{Syl}_p(N_G(C_P)) = \varpi P \).

\( \text{pf.} \) Since \( P \in \text{Syl}_p(G) \), \( p \nmid |G/P| \). Therefore \( p \nmid |N_G(C_P)/P| \).

So \( P \in \text{Syl}_p(N_G(C_P)) \). By the previous theorem (Sylow's 2nd theorem) any Sylow p-subgroup of \( N_G(C_P) \) is a conjugate (in \( N_G(C_P) \)) of \( P \). Since \( P < N_G(C_P) \), we deduce \( \varpi P = \text{Syl}_p(N_G(C_P)) \). \( \Box \)

Corollary. Suppose \( P \in \text{Syl}_p(G) \). Then \( N_G(N_G(C_P)) = N_G(C_P) \).

\( \text{pf.} \) Let \( g \in N_G(N_G(C_P)) \). Then by the previous corollary

\[ \varpi g P g^{-1} = \text{Syl}_p(g N_G(C_P) g^{-1}) \] (Conjugation by \( g \) is an automorphism.)

\[ = \text{Syl}_p(N_G(C_P)) \] (\( g \in N_G(N_G(C_P)) \),)

\[ = \varpi P \] (previous corollary)

\[ \Rightarrow g P g^{-1} = P \Rightarrow g \in N_G(C_P). \]

Therefore \( N_G(N_G(C_P)) \leq N_G(C_P) \). The other direction is clear. \( \Box \)
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(Sylow's 3rd) Theorem. \(| \text{Syl}_p(G) | \equiv 1 \pmod{p} \).

PROOF. Let \( P_o \) be a Sylow \( p \)-subgroup. If \( P_o = \emptyset \), then

\(| \text{Syl}_p(G) | = 1 \) and we are done. So w.l.o.g. we will assume

\( P_o \neq \emptyset \). \( P_o \cap \text{Syl}_p(G) \) by conjugation. So

\(| \text{Syl}_p(G) | = | \text{Syl}_p(G)^{P_o} | ( \pmod{p} ) \). \( \circledast \)

\( P \in \text{Syl}_p(G) \iff \forall \gamma \in P_o, \ P \gamma P_o^{-1} = P \)

\( \iff P_o \subseteq N_G(P) \).

\( \iff P_o \in \text{Syl}_p(N_G(P)) = \emptyset \).

\( \iff P_o = P \).

So \(| \text{Syl}_p(G)^{P_o} | = 1 \).

Therefore by \( \circledast \) \(| \text{Syl}_p(G) | \equiv 1 \).

Sylow's theorems are very instrumental for describing possible group structures of a group with a given order. Here is a standard example:

**Problem.** Describe groups of order \( pq \), where \( p \) and \( q \) are primes, and \( p < q \).
Let \( n_q := |\text{Syl}_q(G)| \); and \( Q_0 \in \text{Syl}_q(G) \).

Since \( G \cap \text{Syl}_q(G) \) transitively, \( |\text{Syl}_q(G)| = |G : Q_0| = [G : N_G(Q_0)] \).

So \( n_q \mid |G/Q_0| \); and, by the 3rd Sylow theorem, \( n_q \equiv 1 \pmod{q} \). Therefore \( n_q \mid p \) and \( q \mid n_q - 1 \).

Since \( p \) is prime, either \( n_q = 1 \) or \( n_q = p \). As \( p < q \) and \( q \mid n_q - 1 \), we get that \( n_q = 1 \); this implies \( Q_0 \triangleleft G \).