Proposition. Suppose $G$ is nilpotent. Then

1. $G$ is solvable.

2. If $1 \neq N \triangleleft G$, then $Z(G) \cap N \neq 1$; in particular
   
   \[ Z(G) \neq 1 \text{ if } G \neq 1. \]

3. $H \leq G$, $N \triangleleft G \implies H$ and $G_i/N$ are nilpotent.

\textbf{Proof.} 1. By induction on $i$, we have $\gamma_i(G) \supseteq G^{(i)}$.

   (In fact using one of your HW assignments you can prove $G^{(i)} \subseteq \gamma_2(G)$.)

2. Suppose $\gamma_i(G) \cap N \neq 1$ and $\gamma_{i+1}(G) \cap N = 1$.

   There is such an $i$ as $\gamma_i(G) \cap N = N \neq 1$ and

   \[ \gamma_{i+1}(G) \cap N = 1. \]

   Then $[\gamma_i(G) \cap N, N] \subseteq \gamma_{i+1}(G) \cap N = 1$.

   So $\gamma_i(G) \cap N \subseteq Z(N)$.

3. By induction on $i$, show $\gamma_i(H) \subseteq \gamma_i(G)$ and

   \[ \gamma_i(G/N) = \gamma_i(G)N/N. \]
Def. The Frattini subgroup \( \Phi(G) \) of a group is the intersection of all of its maximal subgroups.

Observation. \( \forall \Theta \in \text{Aut}(G) \) and \( M \leq G \) maximal, we have 

\[ \Theta(M) \] is a maximal subgroup. So \( \Phi(G) \) is a characteristic subgroup of \( G \).

Theorem. Suppose \( G \) is a finite \( p \)-gp. Then

\[ \Phi(G) = G^p [G,G], \] where \( G^p = \langle g^p \mid g \in G \rangle \).

(Notice that \( G^p \) is not necessarily a subgroup.)

Proof. Suppose \( M \) is a maximal subgroup of \( G \). Since \( G \) is a finite \( p \)-gp, it is nilpotent; and so \( M \triangleleft G \). Therefore \( G/M \) is a group with no proper non-trivial subgroup; therefore \( G/M \) has prime order. Since \( G \) is a \( p \)-gp, \( G/M \cong \mathbb{F}_p \).

Hence \( [G,G] \subseteq M \) as \( G/M \) is abelian; and \( G^p \subseteq M \) as \( G/M \) is \( p \)-torsion; that means \( (gM)^p = M \). Therefore \( \Phi(G) = G^p [G,G] \subseteq M \).

We have proved that \( G^p [G,G] \subseteq \Phi(G) \).
On the other hand, $G/[G[G,G]]$ is an abelian group; and so 
\[ (G/[G,G])^p \] is a normal subgroup. And 
\[ (G/[G,G])^p = \{ g^p \mid g \in G \} \]
\[ = \{ g^p \mid g \in G/G[G,G] = G^p/[G,G] \}. \]
So $G[G,G]$ is a normal subgroup of $G$; and 
$V_0 := G/[G[G,G]]$ is a $p$-torsion abelian group. Hence $G/[G[G,G]]$ is a vector space over the finite field $\mathbb{Z}/p\mathbb{Z}$. (Why?)

Hence for any non-zero vector $v$, there is a subspace $V$ of codimension 1 st. $v \notin V$. Hence $V_0/V \cong \mathbb{Z}/p\mathbb{Z}$ and $v \notin V$. So $V$ is a maximal subgroup of $V_0$.

Suppose $g \in G \setminus G[G,G]$; then $v := g^p[G,G] \in V_0$ and $v \neq 0$.

Now let $V$ be as above. So $V = M/G^p[G,G]$ for some maximal subgroup $M$ of $G$; and $g \notin M$. That means $g \notin \Phi(G)$.

We have proved $g \notin G^p[G,G] \Rightarrow g \notin \Phi(G)$.

So $\Phi(G) \leq G^p[G,G]^2$. 

Claim follows from 1, 2. ■
Here is an alternative way to explain the 2nd part of the proof.

Lemma. Suppose $\Theta: G \to H$ is an onto group homomorphism. Then

$$\Theta(\Phi(G)) \subseteq \Phi(H).$$

Proof. Suppose $M$ is a maximal subgroup of $H$.

Claim. $\Theta^{-1}(M)$ is a maximal subgroup of $G$.

Proof of claim. Since $\Theta$ is a group hom, $\Theta^{-1}(M)$ is a subgroup.

Since $\Theta$ is onto and $M$ is a proper subgroup, $\Theta^{-1}(M)$ is a proper subgroup.

Suppose $\Theta^{-1}(M) \neq \tilde{M} \leq G$.

Subclaim. $\Theta^{-1}(\Theta(\tilde{M})) = \tilde{M}$.

Proof of subclaim. $\tilde{M} \subseteq \Theta^{-1}(\Theta(\tilde{M}))$ is true for any function $\Theta$.

$x \in \Theta^{-1}(\Theta(\tilde{M})) \implies \Theta x \in \Theta(\tilde{M}) \implies \exists \tilde{m} \in \tilde{M}, \Theta x = \Theta \tilde{m}$

$\implies \Theta(\tilde{m}^{-1} x) = 1 \implies \tilde{m}^{-1} x \in \Theta^{-1}(G_{13}) \subseteq \Theta^{-1}(M) \subseteq \tilde{M}$

$x \in \tilde{m} \cdot \tilde{M} = \tilde{M}$. 

Then $\Theta^{-1}(M) \neq \Theta^{-1}(\Theta(\tilde{M}))$. So $M \neq \Theta(\tilde{M}) \leq H$. Since $M$ is max., we deduce that $\Theta(\tilde{M}) = H$ and so $\tilde{M} = \Theta^{-1}(\Theta(\tilde{M})) = G$.
So \( \bigcap_{N < H} \theta^{-1}(M) \supseteq \Phi(G) \). Hence
\[
\theta(\Phi(G)) \subseteq \theta(\bigcap_{N < H} \theta^{-1}(M)) = \bigcap_{M < H \text{ max}} M = \Phi(H).
\]
\[\text{\theta is onto}\]

**Lemma.** Suppose \( V \) is a vector space over \( \mathbb{Z}/p\mathbb{Z} \). Then
\[
\Phi(V) = \{0\}.
\]

(Ex.)

- \( G/\bigcap_{G_1 \leq G} G_1 \) is a vector space over \( \mathbb{Z}/p\mathbb{Z} \) and is trivial.
- \( \pi : G \to G/\bigcap_{G_1 \leq G} G_1 \) is onto. So \( \pi(\Phi(G)) \subseteq \Phi(G/\bigcap_{G_1 \leq G} G_1) \)
\[
\Phi(G) \subseteq \ker \pi = G/\bigcap_{G_1 \leq G} G_1.
\]