\[ \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \text{ where } \bar{g} \text{ means } \\
g \in \mathbb{Z}(\text{SL}_2(\mathbb{R})) \in \text{SL}_2(\mathbb{R})/\mathbb{Z}(\text{SL}_2(\mathbb{R})) =: \text{PSL}_2(\mathbb{R})/\{I, -I\} \]

**Proof.** This time we use the action of \( \text{SL}_2(\mathbb{R}) \) on the upper half plane; \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d} \).

And so \( \begin{bmatrix} 1 & 2n \\ 1 & 1 \end{bmatrix} \cdot z = z + 2n \) and

\[ \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \cdot z = \frac{-1}{z} \]

Therefore \( \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \) sends the blue region to the yellow region, and the yellow region to the blue region.

A shift by at least two steps send \( X_1 \) to \( X_2 \). So we have

\[ \left\langle \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \right\rangle \cdot X_1 \subseteq X_2 \]

\[ \left\langle \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right\rangle \cdot X_2 \subseteq X_1. \]

Thus by the ping-pong lemma

\[ \left\langle \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right\rangle \cong \left\langle \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \right\rangle \ast \left\langle \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right\rangle \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \]
Ex. Suppose \( x > 1 \), and let \( a = \begin{bmatrix} x & \lambda \\ 1 & x^{-1} \end{bmatrix} \). Suppose \( b \in SL_2(\mathbb{R}) \) has the following property:

\[
b \cdot \mathcal{E}_0, \mathcal{E}_\infty \cap \mathcal{E}_0, \mathcal{E}_\infty = \emptyset.
\]

Then

\[
SL_2(\mathbb{R}) \cap \mathbb{R} \cup i\mathbb{R} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}, \quad r := \frac{zx + yt}{zr + t}.
\]

Notice that \( a \) has exactly two fixed points: \( a \cdot 0 = 0, a \cdot \infty = \infty \); and \( a \) is contracting everything except 0 towards \( \infty \).

\[
\text{Fix}(b a b^{-1}) = b \cdot \text{Fix}(a) = \mathcal{E}_0, b \cdot \mathcal{E}_\infty;
\]

and so

\[
\text{Fix}(b a b^{-1}) \cap \text{Fix}(a) = \emptyset.
\]

- \( a \) is contracting everything except 0 towards \( \infty \)
- \( a^{-1} \) is contracting everything except \( \infty \) towards 0
- \( b a b^{-1} \) is contracting everything except \( b \cdot 0 \) towards \( b \cdot \infty \)
- \( b a b^{-1} \) is contracting everything except \( b \cdot \infty \) towards \( b \cdot 0 \)

Let's use circle model of \( \mathbb{R} \cup i\mathbb{R} \).

Suppose \( a_1, a_2 \in \text{Homeo}(S^1) \);

\( a_1 \) has two fixed points \( x^- \) and \( x^+ \). And
there are nbhds $U_x^-$ of $x^-$ and $U_x^+$ of $x^+$ s.t.
\[ a_n^+ \cdot (S^1 \setminus U_x^-) \subseteq U_x^+ \quad \forall n \in \mathbb{Z}^+ \]
\[ a_n^- \cdot (S^1 \setminus U_x^+) \subseteq U_x^- \quad \forall n \in \mathbb{Z}^+ \]

$a_2$ has two fixed points $y^-$ and $y^+$ and there are nbhds $U_y^-$ of $y^-$ and $U_y^+$ of $y^+$ s.t.
\[ a_n^+ \cdot (S^1 \setminus U_y^-) \subseteq U_y^+ \quad \forall n \in \mathbb{Z}^+ \]
\[ a_n^- \cdot (S^1 \setminus U_y^+) \subseteq U_y^- \quad \forall n \in \mathbb{Z}^+ \]

Suppose $U_x^+$ and $U_y^+$'s are disjoint.

Let $X_1 := U_x^+ \cup U_x^-$ and $X_2 := U_y^+ \cup U_y^-$. Then
\[ (\langle a_1 \rangle \setminus I) \cdot X_2 \subseteq X_1 \]
and \( (\langle a_2 \rangle \setminus I) \cdot X_1 \subseteq X_2 \). And so by the ping-pong lemma \( \langle a_1, a_2 \rangle \cong \langle a_1 \rangle \ast \langle a_2 \rangle \cong \mathbb{Z} \ast \mathbb{Z} \). So:

**Theorem.** Suppose $a = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ where $\lambda > 1$; and $b \in SL_2(\mathbb{R}) \setminus \left( \left[ * \ *ight] \cup \left[ * \ *ight] \cup \left[ * \ *ight] \right)$. Then for large enough $n$, \( \langle a^n, b a^n b^{-1} \rangle \cong F_2 \). In particular, \( \langle a, b \rangle \) has a
non-commutative free subgroup.

Remark. The conditions on b are necessary as otherwise <a, b> has a subgroup of index ≤ 2 which is solvable.

Theorem (Jacques Tits) A finitely generated subgroup \( G \) of \( GL_n(F) \) (where \( F \) is a field) is either virtually solvable or it contains a non-commutative free subgroup.

We say \( G \) is virtually solvable if \( G \) has a solvable subgroup of finite index.

In the next lecture we will see that a virtually solvable group does not have a non-commutative free subgroup.