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1. (10 points) Suppose $G$ is a finite group, and $P$ is a Sylow $p$-subgroup. Suppose $|\text{Syl}_p(G)| \geq [G : P]$, where $\text{Syl}_p(G)$ is the set of all the Sylow $p$-subgroups of $G$. Prove that for any $g \in G$ we have $g \in (P, gPg^{-1})$.

By Sylow’s 2nd theorem, $|\text{Syl}_p(G)| = [G : N_G(P)] \cdot |G : P| \geq |\text{Syl}_p(G)|$.

Since $P \leq N_G(P)$, we have $|G : P| \geq |G : N_G(P)|$.

By assumption we have $|\text{Syl}_p(G)| \geq [G : P]$.

$\text{1, 2} \implies |\text{Syl}_p(G)| = [G : P]$. And so $[G : P] = [G : N_G(P)]$.

And so $|P| = |N_G(P)| \
\implies P = N_G(P)$.

Let $H_g := \langle P, gPg^{-1} \rangle$. Then $P$ and $gPg^{-1}$ are Sylow $p$-subgroups of $H_g$. ($|P| = |gPg^{-1}|$ is the largest power of $p$ which divides $|G|$ and $|H_g| > |G|$; so $|P|$ is the largest power of $p$ which divides $|H_g|$.) Hence by Sylow’s 2nd theorem, $\exists h \in H_g$ s.t. $hP h^{-1} = gPg^{-1}$. And so $(h^{-1} g) P (h^{-1} g) = P$; this means $h^{-1} g \in N_G(P) = P$. Therefore $g \in hP \subseteq H_g$. 

\[ \text{Page 2} \]
2. Suppose $G$ is a non-abelian finite group, $Z(G)$ is its center, and $G/Z(G)$ is a $p$-group.

(a) (5 points) Prove that $G$ has a unique Sylow $p$-subgroup $P$. (Hint: Think about $[G : N_G(P)Z(G)]$.)

$$\forall g \in Z(G), \ gPg^{-1} = P.$$ So $Z(G) \leq N_G(P)$; this implies $Z(G) N_G(P) = N_G(P)$. So 


Notice, since $P$ is a Sylow $p$-subgp, $\gcd([G : P], p) = 1$. Therefore $\gcd([G : N_G(P)], p) = 1$. On the other hand,

$$[G : N_G(P)] = [G/Z(G) : N_G(P)/Z(G)] \Bigg| \frac{|G/Z(G)|}{|G/Z(G)|} \Rightarrow [G : N_G(P)]$$

is a power of $p$ should be a power of $p$ $\Rightarrow 1$, 2 imply that $[G : N_G(P)] = 1$; this implies $G = N_G(P)$, and so $P \triangleleft G$. 

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(b) (5 points) Prove that \( p | |Z(G)| \).

**Solution 1.** \( G \vartriangleleft G \) by conjugation. \( Z(G) \) is the kernel of this action. So we get an action \( G/Z(G) \rightarrow G \cdot \\
(gZ(G)) \cdot g' = g \cdot g \cdot g^{-1} \).

Since \( G/Z(G) \) is a \( p \)-gp, we have

\[ |G| \equiv |\text{fixed points of } G/Z(G)| \pmod{p}. \]

The set of fixed points of \( G/Z(G) = \{ g' \in G | \forall g \in Z(G), g Z(G) \cdot g = g' \} \)

\[ = \{ g' \in G | \forall g \in G, g \cdot g \cdot g^{-1} = g' \} = Z(G). \]

Hence \( |G| \equiv |Z(G)| \pmod{p} \). As \( p \mid |G| \), we get that \( p \mid |Z(G)| \).

**Solution 2.** Suppose to the contrary that \( p \nmid |Z(G)| \). So

\( |G| = |G/Z(G)| \cdot |Z(G)| \) implies that \( |G/Z(G)| \) is the largest power of \( p \) which divides \( |G| \). And so \( |P| = |G/Z(G)| \).

On the other hand, \( P, Z(G) \vartriangleleft G \) and \( \gcd(|P|, |Z(G)|) = 1 \). Hence

\( \Phi(Z(G)) = 1 \) as \( |P \cdot Z(G)| \mid |P| \) and \( |P \cdot Z(G)| \mid |Z(G)| \). And so

\( |P \cdot Z(G)| = |P| \cdot |Z(G)| = |G| \). Therefore \( G = P \cdot Z(G) \) \( \Box \).

Since \( P \) is a finite \( p \)-gp, \( Z(P) \neq 1 \). Suppose \( g \in P \setminus \{1\} \). Then by \( \Box \), \( \forall g \in G \), \( \exists g_p \in P \) and \( z \in Z(G) \) s.t. \( g = g_p \cdot z \). And so

\[ g \cdot z = g_p \cdot z \cdot g_p = g_p \cdot g \cdot g_p \cdot z = g_p \cdot g \cdot g_p \cdot z. \]

This implies \( g \in P \cdot Z(G) \) which is a contradiction.
3. Suppose $G$ is a group of order 56. Let $P_2$ be a Sylow 2-subgroup of $G$, and $P_7$ be a Sylow 7-subgroup of $G$.

(a) (5 points) Prove that either $P_2$ is normal in $G$ or $P_7$ is normal in $G$.

Let $n_2 = |\text{Syl}_2(G)|$. By Sylow theorems we know: $n_2 = \frac{|G|}{|N_G(P_2)|}$ and $n_2 \equiv 1 \pmod{2}$. Looking at the set $\{1, 2, 4, 8\}$ of positive divisors of 8, we see that the only possibilities for $n_2$ are 1 and 8. If $n_2 = 1$, then there is a unique Sylow 2-subgroup $P_2$ and so $P_2 \triangleleft G$. Suppose $n_2 = 8$; and $P_7^{(1)}, \ldots, P_7^{(8)}$ are distinct Sylow 7-subgroups. If $i \neq j$, then $|P_7^{(i)} \cap P_7^{(j)}| = 1$ and it is not 7. So if $i \neq j$, then $P_7^{(i)} \cap P_7^{(j)}$. We also notice that, $\forall g \in P_7 \setminus \{1\}$, $\langle g \rangle = 7$ as $1 \nmid |\langle g \rangle| \mid 7$. On the other hand, $\langle g \rangle = 7 \Rightarrow |\langle g \rangle| = 7 \Rightarrow \langle g \rangle$ is a Sylow 7-subgroup of $G$ ($7 \mid |G|$ and $7^2 \nmid |G|$). So $\langle g \rangle = P_7^{(i)}$ for some $i$. Hence $\{g \in G \mid \langle g \rangle \neq 7\} = G \setminus \left( \bigcup_{i=1}^{8} (P_7^{(i)} \setminus \{1\}) \right)$. Thus

\[ |\{g \in G \mid \langle g \rangle \neq 7\}| = |G| - 8 \times 6 = 56 - 48 = 8 \]

\[ = |P_2| \quad \text{(1)} \]

\[ (\forall g \in P_2 \Rightarrow \langle g \rangle \neq 7 \Rightarrow \langle g \rangle \neq 7) \Rightarrow P_2 \subseteq \{g \in G \mid \langle g \rangle \neq 7\} \quad \text{(2)} \]

(1, 2) imply that $P_2 = \{g \in G \mid \langle g \rangle \neq 7\}$. Since conjugation does not change order, we have that, $\forall g' \in G$, $g' \{g \in G \mid \langle g \rangle \neq 7\}^{-1} g^{-1} = \{g \in G \mid \langle g \rangle \neq 7\}$

And so $P_2 \triangleleft G$.
(b) (5 points) Show that there are at least two non-isomorphic non-abelian groups of order 56. (Hint: You are allowed to use without proof that $|\text{GL}_3(\mathbb{Z}/2\mathbb{Z})| = (7)(24).$)

Claim: \exists a non-trivial group homomorphism $c: \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(\mathbb{Z}/7\mathbb{Z})$;

\[ |\text{Aut}(\mathbb{Z}/7\mathbb{Z})| = \varphi(7) = 6 \quad \text{and} \quad 2 \mid 6. \] So by Cauchy's theorem, \exists an element of order 2 in $\text{Aut}(\mathbb{Z}/7\mathbb{Z})$.

\[ c(1+2\mathbb{Z})(x) = -x. \]

So there are non-trivial group homomorphisms $c_1: \mathbb{Z}/8\mathbb{Z} \to \text{Aut}(\mathbb{Z}/7\mathbb{Z})$, $c_2: \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(\mathbb{Z}/7\mathbb{Z})$, $c_3: (\mathbb{Z}/2\mathbb{Z})^3 \to \text{Aut}(\mathbb{Z}/7\mathbb{Z})$. To get $c_i$'s it is enough to look at the following composite group hom:

$\mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{c} \text{Aut}(\mathbb{Z}/7\mathbb{Z})$, $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$

$(x, y) \mapsto y$

$(\mathbb{Z}/2\mathbb{Z})^3 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{c} \text{Aut}(\mathbb{Z}/7\mathbb{Z})$

$(x_1, x_2, x_3) \mapsto x_1$

Since $c_i$'s are non-trivial, $(\mathbb{Z}/8\mathbb{Z}) \times c_1(\mathbb{Z}/7\mathbb{Z})$, $(\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \times c_2(\mathbb{Z}/7\mathbb{Z})$, $(\mathbb{Z}/2\mathbb{Z})^3 \times c_3(\mathbb{Z}/7\mathbb{Z})$ are all non-abelian (Notice that in $H \times_f N$, $h \cdot n^{-1} = f(cn)$ when $H$, $n \mathbb{N}$. So, if $f$ is non-trivial, $H$ does NOT commute with $N$; and so $H \not\subset H \times_f N$.) And Sylow 2-subgps of these three groups are not isomorphic.

You could use $|\text{Aut}((\mathbb{Z}/2\mathbb{Z})^3)| = |\text{GL}_3(\mathbb{Z}/2\mathbb{Z})| = (7)(24)$ and Cauchy's theorem to find a non-trivial group hom. $c: \mathbb{Z}/7\mathbb{Z} \to \text{Aut}((\mathbb{Z}/2\mathbb{Z})^3)$; and consider $c(\mathbb{Z}/7\mathbb{Z}) \times c((\mathbb{Z}/2\mathbb{Z})^3)$. And notice that here there is a unique Sylow 2-subgp.

Good Luck!