Homework 8

Saturday, December 1, 2018 11:32 AM

1. Suppose R is a unital ring that is not necessarily commutative. Prove that $\tilde{I}$ is a both sided ideal of $M_n(R)$ if and only if $\tilde{I} = M_n(I)$ for some both sided ideal I of R.

(Remark. Proof of this result just needs a bit of patience with matrix computations, but it is an important result. In particular, it shows that $M_n(\mathbb{C})$ has no proper non-zero both sideded ideal.)

(Hint. Let $e_{ij} \in M_n(R)$ s.t. the i,j entry of e_{ij} is 1 and the other entries are 0. Then $e_{ij}\, e_{kl} = \begin{cases} 0 & \text{if } j \neq k, \\ e_{il} & \text{if } j = k. \end{cases}$

So, for $a = [a_{ij}]$, $e_{kk}\, a\, e_{ll} = \sum_{i,j} a_{ij}\, e_{kk} e_{ij} e_{ll}$

$= a_{kl}\, e_{kl}$.

- Let $I := \{ x \in R \mid x \text{ is an entry of an element of } \tilde{I} \}$.)

2. Suppose R is a unital commutative ring.

(a) For $\mathfrak{p} \in \operatorname{Spec}(R)$, let $\mathfrak{p}[x] := \{ \sum_{i=0}^{n} a_i x^i \in R[x] \mid a_i \in \mathfrak{p} \}$. $n \in \mathbb{Z}^{\geq 0}$

Prove that $\mathfrak{p}[x] \in \operatorname{Spec}(R[x])$ (Hint. Use $R[x] \to (R/\mathfrak{p})[x]$, $\sum a_i x^i \mapsto \sum (a_i + \mathfrak{p}) x^i$.)

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(b) Prove that $Nil(R[x]) = Nil(R)[x]$, where again

$$Nil(R)[x] := \left\{ \sum_{i=0}^{n} a_i x^i \in R[x] \,\middle|\, a_i \in Nil(R),\ n \in \mathbb{Z}^{\geq 0} \right\}.$$

(c) Prove that

$$R[x]^{\times} = \{ a_0 + a_1 x + \cdots + a_n x^n \mid a_0 \in R^{\times}, a_1, \ldots, a_n \in Nil(R), n \in \mathbb{Z}^{+} \}.$$

3. Convience yourself that $\mathbb{Q}[\sqrt{2}] = \{ a + \sqrt{2}\, b \mid a, b \in \mathbb{Q} \}$ is a subring of $\mathbb{R}$. Let $\phi : \mathbb{Q}[\sqrt{2}] \longrightarrow M_2(\mathbb{Q})$,

$$\phi(a + \sqrt{2}\, b) := \begin{bmatrix} a & b \\ 2b & a \end{bmatrix}.$$

Prove that ϕ is a ring homomorphism, and deduce that

$$\mathbb{Q}[\sqrt{2}] \simeq \left\{ \begin{bmatrix} a & b \\ 2b & a \end{bmatrix} \,\middle|\, a, b \in \mathbb{Q} \right\}.$$

4. Let I be the ideal generated by 2 and x in $\mathbb{Z}[x]$. Prove that I is not a principal ideal.

5. Let $\omega := \frac{-1 + i\sqrt{3}}{2}$ and $\mathbb{Z}[\omega] = \{ a + b\omega \mid a, b \in \mathbb{Z} \}$. Convience yourself that $\mathbb{Z}[\omega]$ is a subring of $\mathbb{C}$.

(a) Let $N(a + b\omega) := (a + b\omega)(a + b\overline{\omega})$. Convience yourself that $N(z_1 z_2) = N(z_1) N(z_2)$ and $N(a + \omega b) = a^2 - ab + b^2$.

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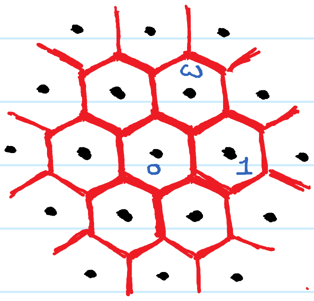
Prove that $\forall z_1, z_2 \in \mathbb{Z}[\omega]$, $z_2 \neq 0$, $\exists q, r \in \mathbb{Z}[\omega]$ s.t.

$$z_1 = z_2 \cdot q + r \quad \text{and} \quad N(r) < N(z_2).$$

(b) Prove that $\mathbb{Z}[\omega]$ is a PID.

(c) Prove that $\mathbb{Z}[\omega]^{\times} = \{\pm 1, \pm\omega, \pm\omega^2\}$.

(This problem does not need a help, but the following picture can be helpful:



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6. Let n be a square-free integer more than 3. Let

$$\mathbb{Z}[\sqrt{-n}] = \{a + \sqrt{-n}\, b \mid a, b \in \mathbb{Z}\}.$$

(a) Prove that $2, \sqrt{-n}, 1 \pm \sqrt{-n}$ are all irreducible in $\mathbb{Z}[\sqrt{-n}]$.

(b) Find an element in R which is irreducible and not prime

(c) Show that $\mathbb{Z}[\sqrt{-n}]$ is not a UFD.