In the previous lecture we showed \( G / G_x \to G \times, gG_x \mapsto g \circ x \) is a bijection, and \( G_x := \{ g \circ x \mid x \in X \} \) is a partition of \( X \).

**Lemma.** \( \forall g \in G, x \in X, G_{g \circ x} = g \circ G_x \circ g^{-1} \); in particular if \( |G_x| < \infty \), then \( |G_{g \circ x}| = |G_x| \).

**Pf.** \( h \in G_{g \circ x} \iff h \circ (g \circ x) = g \circ x \)

\( \iff (g^{-1} \circ h) \circ x = x \)

\( \iff g^{-1} \circ h \in G_x \iff h \in g \circ G_x \circ g^{-1} \).

**Def.** Suppose \( G \triangleleft X \); \( \forall g \in G, X^g := \{ x \in X \mid g \circ x = x \} \)

(the set of fixed points of \( g \)).

**Lemma.** \( \forall g, h \in G, g \circ X^h = X^{ghg^{-1}} \).

**Pf.** \( x \in X^h \iff (ghg^{-1}) \circ x = x \)

\( \iff h \circ (g^{-1} \circ x) = g^{-1} \circ x \)

\( \iff g^{-1} \circ x \in X^h \iff x \in g \circ X^h \).

**Cor.** \( |X^h| = |X^{ghg^{-1}}| \quad \forall g, h \in G. \)
Theorem. \( G \cap X, |G|, |X| < \infty \). Then

\[
|G \cdot x| = \frac{1}{|G|} \sum_{g \in G} |x^g|
\]

(The number of elements of the quotient space is the average of the number of fixed points of elements of \( G \)).

\[\text{Proof.} \quad A := \frac{1}{2} \sum_{(g,x) \in G \times X} 1_{g \cdot x = x^g}. \text{ Then} \]

\[
|A| = \sum_{g \in G} |x^g| = \sum_{x \in X} |G \cdot x|.
\]

\[
= \sum_{G \cdot x \in G \times X} \sum_{y \in G \cdot x} |G_y|.
\]

Since \( y \in G \cdot x \), \( G_y \) is a conjugate of \( G \cdot x \),

\[
= \sum_{G \cdot x \in G \times X} \sum_{y \in G \cdot x} |G_x| |G \cdot x|.
\]

\[\text{Orbit-Stabilizer theorem} \quad \Rightarrow \sum_{G \cdot x \in G \times X} |G|.
\]

\[\Rightarrow \sum_{G \cdot x \in G \times X} |G| |G \cdot x|; \text{ and claim follows.} \]

Def. We say \( G \cap X \) is transitive if \( X = G \cdot x \).
Proposition. Suppose \( G \acts X \) is transitive and \( |G| < \infty, |X| > 1 \).

Then \( \exists g \in G \) s.t. \( X^g = \emptyset \).

Proof. Suppose to the contrary that \( X^g \neq \emptyset \) for any \( g \in G \). Hence 
\[
|X^g| \geq 1 \quad \forall g \in G.
\]
So by Lemma that is not Burnside's we have 
\[
|G^X| = \frac{1}{|G|} \sum_{g \in G} |X^g| \\
= \frac{1}{|G|} \left( |X| + \sum_{g \in G \setminus \{e\}} |X^g| \right) \\
\geq \frac{1}{|G|} \left( |X| + |G| - 1 \right).
\]
Since \( G \acts X \) is transitive, \( |G^X| = 1 \); and so 
\[
1 \geq \frac{1}{|G|} \left( |X| + |G| - 1 \right) \quad \text{which implies} \quad |X| \leq 1;
\]
and that is a contradiction. \( \blacksquare \)

Problem. Suppose \( G \) is a finite group, \( H \triangleleft G \). Prove that 
\[
G \neq \bigcup_{g \in G} gHg^{-1}.
\]

Solution. \( G \acts G/H \) by left translations transitively \( \Rightarrow \)
\[
\exists g \in G, \quad (G/H)^g = \emptyset.
\]
\((G/H)^* = \emptyset \implies g_0 \notin \bigcup_{x \in X} G_x\). 

\(g' \in \text{The stabilizer of } gH \iff g'gH = gH\)

\[\iff g'^{-1}g'g \in H\]

\[\iff g' \in gHg^{-1}\]

So stabilizer of \(gH = gHg^{-1}\). Hence \((*)\) implies

\[g_0 \notin \bigcup_{g \in G} gHg^{-1}\].

**Def.** \(X^G := \{x \in X \mid \forall g \in G, \ g \cdot x = x\} \). (The set of fixed points of \(G\)).

Notice that \(x \in X^G \iff |G \cdot x| = 1\). So

\[|X| = \sum_{G \cdot x \in X^G} |G \cdot x| = |X^G| + \sum_{G \cdot x \in X \setminus X^G} |G \cdot x| \]

\[\implies |X| = |X^G| + \sum_{G \cdot x \in X \setminus X^G} [G : G_x] \text{ by Orbit-Stabilizer.}\]
Let's study $G \cong G$ by conjugation.

$G$-orbit of $g$ is $\{g'gg'^{-1} \mid g' \in G\}$; this is called the conjugacy class of $g$, and we denote it by $\text{Cl}(g)$.

The stabilizer of $g = \{g' \in G \mid g'g^{-1}g = g\}$

$= \{g' \in G \mid g' = gg'\}$

$= C_G(g)$ is called the centralizer of $g$.

So $|\text{Cl}(g)| = [G : C_G(g)]$ by orbit-stabilizer theorem.

The set of fixed points of $G = \{g \in G \mid \forall g' \in G, g'gg'^{-1} = g\}$

$= \{g \in G \mid \forall g' \in G, g' = gg'\}$

$= Z(G)$ is called the center of $G$.

By the previous equation, we have

$$|G| = |Z(G)| + \sum_{\text{representative of conjugacy classes; not in } Z(G)} [G : C_G(g)]$$

This is called the class equation.
Def. Suppose $G \curvearrowright X$. Kernel of this action is

$$\exists g \in G \mid \forall x \in X, g \cdot x = x\cdot g.$$ 

Important. If $f : G \rightarrow S_X$ is the group homomorphism associated with $G \curvearrowright X$, then the kernel of the group action is $\ker(f)$; and so it is a normal subgroup of $G$ and by the 1st isomorphism theorem $G/\ker f \cong S_X$.

Let's study $G \curvearrowright G/H$ by the left translations.

- Action is transitive.
- Stabilizer of $gH$ is $gHg^{-1}$.
- Kernel of this action is $\bigcap_{g \in G} gHg^{-1}$. This is called the normal core of $H$, and we denote it by $\text{cor}(H)$.

Hence $G/\text{cor}(H) \hookrightarrow S(G/H)$; in particular,

$$[G: \text{cor}(H)] \mid [G:H]!.$$ 

Lemma. Suppose $H \leq G$, $N \triangleleft G$, and $N \leq H$. Then $N \leq \text{cor}(H)$.
Lecture 03: Normal core of a subgroup

Thursday, October 4, 2018 12:54 AM

pf. \( N \subseteq H \Rightarrow gNg^{-1} \subseteq gHg^{-1} \)

\[
N \subseteq gHg^{-1} \Rightarrow N \subseteq \bigcap_{g \in G} gHg^{-1} = \text{cor}(H).
\]

So \( \text{cor}(H) \) is the largest normal subgroup of \( G \) that is contained in \( H \).

Problem. Suppose \( G \) is a finite group, \( H \leq G \), \( [G:H] = p \), where \( p \) is the smallest prime factor of \( |G| \). Prove that \( H < G \).

Solution. By the previous discussion,

\[
[G:\text{cor}(H)] \mid [G:H]! = p!.
\]

And \( [G:\text{cor}(H)] \mid |G| \). Hence \( [G:\text{cor}(H)] \mid \gcd(|G|, p!) \)

Since \( p \) is the smallest prime factor of \( |G| \), \( \gcd(|G|, p!) = p \).

Therefore \( [G:\text{cor}(H)] \mid p \), which implies \( [G:\text{cor}(H)] = p \).

As \( \text{cor}(H) \subseteq H \) and \( [G:H] = p = [G:\text{cor}(H)] \), we deduce \( H = \text{cor}(H) < G \).
G \triangleleft H \iff H \leq G^g \text{ by conjugation.} \\

\text{Stab. of } H = \{ g \in G \mid gHg^{-1} = H \} \\

\text{is called the normalizer of } H \text{ in } G \\

\text{and it is denoted by } N_G(H). \\

\text{Orbit of } H = \{ gHg^{-1} \mid g \in G \} \\

\text{So } \# \text{ of conjugates of } H = [G : N_G(H)]. \\

\text{Notice that } N_G(H) \text{ is the largest subgroup of } G \text{ which has } H \text{ as a normal subgroup.} \\

\text{Next we prove an extremely useful result about actions of } p\text{-groups.} \\

\textbf{Theorem.} \text{ Suppose } |G| = p^n \text{ where } p \text{ is prime. And } G \triangleleft X, |X| < \infty. \text{ Then } |X| \equiv |X^G| \pmod{p}. \\

\text{We will prove this in the next lecture, and then use it to prove Sylow theorems.}