Lecture 06: Exact sequences

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In the previous lecture we learned how Sylow theorems can help us find non-trivial normal subgroups. How can we go further? More precisely we would like to know having N and G/N if we can parametrize all the possibilities of G. This is a hard question; but we will learn some tools to deal with this problem.

<u>Def.</u> Suppose $G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} G_n$ is a sequence of groups and group homomorphisms. This is called <u>an exact sequence</u> if $\operatorname{Im}\phi_i = \ker \phi_{i+1}$ for $1 \le i \le n-1$.

- An exact sequence of the form $1 \to G_1 \to G_2 \to G_3 \to 1$ is called a <u>Short Exact Sequence</u> (SES).

- We say the SES $1 \to G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} G_3 \to 1$ <u>splits</u> if $\exists\, \psi: G_3 \to G_2$ s.t. $\phi_2 \circ \psi = id_{G_3}$, and we write

$$1 \to G_1 \xrightarrow{\phi_1} G_2 \underset{\psi}{\overset{\phi_2}{\rightleftarrows}} G_3 \to 1.$$

is a commutative diagram.

- We say $(\theta_1, \theta_2, \theta_3)$ is a SES homomorphism if

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$$\begin{tikzcd} 1 \arrow[r] & G_1 \arrow[r, "\phi_1"] \arrow[d, "\theta_1"'] \arrow[dr, phantom, "\circlearrowleft"] & G_2 \arrow[r, "\phi_2"] \arrow[d, "\theta_2"'] \arrow[dr, phantom, "\circlearrowleft"] & G_3 \arrow[r] \arrow[d, "\theta_3"'] & 1 \\ 1 \arrow[r] & G_1' \arrow[r] & G_2' \arrow[r] & G_3' \arrow[r] & 1 \end{tikzcd}$$

is a commutative diagram (this means following arrows gives you the same value no matter which path you take.) ; and each row is a SES.

Similarly we can define an isomorphism $(\theta_1, \theta_2, \theta_3)$ of SES.

A few observations:

- $1 \to H \xrightarrow{\phi} G$ is an exact sequence
  $$\Updownarrow$$
  $\ker \phi = 1 \Leftrightarrow \phi$ is injective.
- $G \xrightarrow{\phi} K \xrightarrow{\phi'} 1$ is an exact sequence
  $$\Updownarrow$$
  $\operatorname{Im} \phi = \ker \phi' = K \Leftrightarrow \phi$ is surjective.

<u>Lemma</u>. Suppose $1 \to G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} G_3 \to 1$ is a SES. Then $\phi_1(G_1) \triangleleft G_2$ and the above SES is isomorphic to

$$1 \to \phi_1(G_1) \xhookrightarrow{i} G_2 \xrightarrow{\pi} G_2/\phi_1(G_1) \to 1$$

where $i(g) = g$ and $\pi(g) = g\,\phi_1(G_1)$.

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Pf. $\Phi_1(G_1) = \ker \Phi_2 \Rightarrow \Phi_1(G_1) \trianglelefteq G_2$. By the 1^{st} isomorph. theorem $\bar{\Phi}_2 : G_2/\ker\Phi_2 \xrightarrow{\sim} \operatorname{Im}\Phi_2$,

$$\bar{\Phi}_2(g \ker \Phi_2) := \Phi_2(g).$$

And $\operatorname{Im}\Phi_2 = G_3$.

$$\begin{tikzcd}
1 \arrow[r] & G_1 \arrow[r] \arrow[d, "\Phi_1"', "\wr"] & G_2 \arrow[r] \arrow[d, "\mathrm{id}_{G_2}"', "\wr"] & G_3 \arrow[r] \arrow[d, "\bar{\Phi}_2^{-1}"', "\wr"] & 1 \\
1 \arrow[r] & \Phi_1(G_1) \arrow[r] & G_2 \arrow[r] & G_2/\Phi_1(G_1) \arrow[r] & 1
\end{tikzcd}$$

$g_1 \mapsto \Phi_1(g_1)$, $\Phi_1(g_1) \mapsto \Phi_1(g_1)$, $g_2 \mapsto \Phi_2(g_2)$, $g_2 \mapsto g_2\Phi_1(G_1)$

(Φ_1 is injective) (by the above argument)

∎

Proposition. A S.E.S. $1 \to N \to G \to G/N \to 1$ splits if and only if $\exists\, H \leq G$ s.t. $H \cap N = 1$ and $HN = G$; in this case $H \simeq G/N$.

Pf. ($\Rightarrow$) Suppose $1 \to N \xrightarrow{i} G \xrightarrow{\pi} G/N \to 1$ splits. Then $\exists\, \psi : G/N \to G$ s.t. $\pi \circ \psi = \mathrm{id}_{G/N}$; this means for any $g \in G$, $\psi(gN)\,N = gN$ (1)

Let $H := \psi(G/N) \leq G$. We show that H has the desired

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properties.

- $H \cap N = 1$. Suppose $g \in H \cap N$. Then $\exists g' \in G$ s.t. $g = \psi(g'N)$. By (1), $\psi(g'N)N = g'N$. Hence $gN = g'N$; as $g \in N$, we deduce $g' \in N$. Therefore $g = \psi(g'N) = \psi(N) = 1$.

- $G = HN$. Suppose $g \in G$. Then by (1)

$$g \in \psi(gN)\, N \subseteq HN; \text{ and so } G = HN.$$

Notice. $$H/_{H\cap N} \xrightarrow{\sim} HN/_N$$
$$h(H\cap N) \longmapsto hN$$

Pf. Let $\pi: G \to G/_N$ be the natural quotient map. By the 1^{st} isomorphism theorem applied to $\pi|_H : H \to G/_N$, we get $\overline{\pi|_H} : H/_{\ker(\pi|_H)} \to \mathrm{Im}(\pi|_H)$, $\overline{\pi|}_H(h \ker(\pi|_H)) := \pi|_H(h) = hN$ is an isomorphism; $\ker(\pi|_H) = H \cap \ker \pi = H \cap N$; $\mathrm{Im}(\pi|_H) = \pi(H) = HN/_N$. $\square$

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By the above notice, $G/N = HN/N \simeq H/H\cap N \simeq H$.

($\Leftarrow$) By the above argument

$\bar{\pi}: H \to G/N$, $\bar{\pi}(h) := hN$ is an isomorphism

when $H\cap N = 1$ and $G = HN$. Let $\psi := (\bar{\pi})^{-1}$; and so

$\psi: G/N \xrightarrow{\sim} H$, $\psi(hN) = h$ for any $h\in H$.

Since $G=HN$, for $g\in G$, $\exists\, h\in H$ s.t. $gN = hN$.

Hence $\psi(gN)\,N = \psi(hN)\,N = hN = gN$.

Therefore $\pi\circ\psi = \mathrm{id}_{G/N}$; and $1\to N\to G\to G/N\to 1$

splits. ■

Next we see how one can describe possible group structures

of G if $1\to N\to G\to H\to 1$ is a split S.E.S. and N

and H are given. Motivated by the previous proposition,

we consider the following case:

$N \trianglelefteq G$, $H\le G$, $H\cap N = 1$, $HN = G$.

<u>Claim</u> $H\times N \xrightarrow{\theta} G$, $(h,n)\mapsto hn$ is a bijection.

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<u>pf of claim.</u> Since $HN=G$, θ is surjective.

$$\theta(h_1,n_1)=\theta(h_2,n_2) \Rightarrow h_1n_1=h_2n_2 \Rightarrow h_1^{-1}h_2=n_1n_2^{-1}\in H\cap N$$

$$\Rightarrow h_1^{-1}h_2=1 \text{ and } n_1n_2^{-1}=1 \Rightarrow h_1=h_2 \text{ and } n_1=n_2;$$

and so θ is injective. $\square$

So as a set G can be identified with $H\times N$. Can we understand multiplication of G under this identification?

$$\begin{array}{ccccc} (h_1,n_1) & \cdot & (h_2,n_2) & := & (h_1h_2,\ c(h_2^{-1})(n_1)\,n_2) \\ \downarrow & & \downarrow & & \downarrow \\ h_1n_1 & \cdot & h_2n_2 & = & \underbrace{h_1h_2}_{\text{in } H}\cdot\underbrace{\underbrace{h_2^{-1}n_1h_2}_{\text{in } N}\cdot n_2}_{\text{in } N} \end{array}$$

$$c: H\to \mathrm{Aut}(N),\ c(h)(n):=hnh^{-1}.$$

<u>Def. / Prop.</u> For $f\in \mathrm{Hom}(H,\mathrm{Aut}(N))$, let

$(h_1,n_1)\cdot(h_2,n_2):=(h_1h_2,\ f(h_2^{-1})(n_1)\,n_2)$. Then $(H\times N,\cdot)$ forms a group. It is called a semi-direct product of H and N. It is denoted by $H\ltimes_f N$ or simply $H\ltimes N$.

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Notice

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] & H \ltimes_f N \arrow[r, shift left] & H \arrow[r] \arrow[l, shift left] & 1 \end{tikzcd}$$

$(h, n) \mapsto h$

$n \mapsto (1, n)$

$(h, 1) \leftarrow\!\shortmid h$

is a split S.E.S.; and the above argument gives us

<u>Theorem</u>. Suppose $1 \to N \to G \to H \to 1$ is a split S.E.S.. Then $\exists f \in \mathrm{Hom}(H, \mathrm{Aut}\, N)$, θ s.t.

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & G \arrow[r] \arrow[d, "\wr\theta"] \arrow[dr, phantom, "\circlearrowright"] & H \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & N \arrow[r] & H \ltimes_f N \arrow[r] & H \arrow[r] & 1 \end{tikzcd}$$

<u>Remark</u>. Suppose G is finite, $N \trianglelefteq G$, $H \leq G$, $N \cap H = 1$.

Then it is enough to know $|H| = |G/N|$ to deduce that G is isomorphic to a semi-direct product of H and N. <u>Why</u>?

$$|HN/N| = |H/H \cap N| = |H| = |G/N| \Rightarrow |G| = |HN|$$

$\Rightarrow G = HN$. Hence $1 \to N \to G \to G/N \to 1$ splits.

$$G/N \cong H$$

And so $G \simeq H \ltimes_f N$ for some f.