Lecture 08: Product of disjoint permutations

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At the end of the previous lecture we stated the following lemma:

Lemma. Suppose $\tau_i \in S_m$, $\operatorname{supp}\tau_i \cap \operatorname{supp}\tau_j = \emptyset$ if $i \neq j$.

Then $(\tau_1 \cdots \tau_n)|_{\operatorname{Supp}\tau_i} = \tau_i|_{\operatorname{Supp}\tau_i}$; and

$$\operatorname{Supp}(\tau_1 \cdots \tau_n) = \bigcup_{i=1}^{n} \operatorname{supp}\tau_i .$$

In particular $(\tau_1 \cdots \tau_n)|_{\operatorname{Supp}\tau_i} \in S_{\operatorname{Supp}\tau_i}$.

Pf $\operatorname{Supp}\tau_i \subseteq \bigcup_{j\neq i} \operatorname{Fix}\tau_j$ as τ_j's are disjoint.

Since $\tau_i(\operatorname{Supp}\tau_i) = \operatorname{Supp}\tau_i$, $\forall x \in \operatorname{Supp}\tau_i$

$$\tau_i(\tau_{i+1} \cdots \tau_n)(x) = \tau_i(x) \text{ and } \tau_1 \cdots \tau_{i-1}(\tau_i(x)) = \tau_i(x).$$

And so $\tau_1 \cdots \tau_n(x) = \tau_i(x)$. Therefore $\operatorname{supp}\tau_i \subseteq \operatorname{Supp}\tau_1 \cdots \tau_n$

Thus $\operatorname{Supp}\tau_1 \cdots \tau_n = \bigcup_{i=1}^{n} \operatorname{Supp}\tau_i$.

We also deduce $(\tau_1 \cdots \tau_n)(\operatorname{Supp}\tau_i) = \tau_i(\operatorname{Supp}\tau_i)$

$$= \operatorname{Supp}\tau_i ;$$

and so $(\tau_1 \cdots \tau_n)|_{\operatorname{Supp}\tau_i} \in S_{\operatorname{Supp}\tau_i}$. ■

Corollary. Suppose $\operatorname{supp}\tau_i \cap \operatorname{Supp}\tau_j = \emptyset$ if $i \neq j$, $X \subseteq [1..n]$,

and $|X| \geq 2$. Then X is an orbit of $\langle \tau_1 \cdots \tau_n \rangle$ if and only if

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X is an orbit of τ_i for some i.

<u>Pf.</u> ($\Rightarrow$) Suppose X is the orbit of $x \in X$. Since $|X| \geq 2$,

$x \in \operatorname{Supp} \tau_1 \cdots \tau_n \underset{\text{by the previous Lemma}}{=} \bigsqcup_{i=1}^{n} \operatorname{supp} \tau_i$. So $\exists\, i$ s.t. $x \in \operatorname{Supp} \tau_i$.

Again by the previous lemma $\langle (\tau_1 \cdots \tau_n)|_{\operatorname{Supp}\tau_i} \rangle = \langle \tau_i|_{\operatorname{Supp}\tau_i} \rangle$.

Hence $\langle \tau_1 \cdots \tau_n \rangle \cdot x = \langle \tau_i \rangle \cdot x$; and claim follows.

($\Leftarrow$) Suppose $X = \langle \tau_i \rangle \cdot x$. Then $x \in \operatorname{Supp} \tau_i$; again by the previous lemma $\langle (\tau_1 \cdots \tau_n)|_{\operatorname{Supp}\tau_i} \rangle = \langle \tau_i|_{\operatorname{Supp}\tau_i} \rangle$; and so $\langle \tau_i \rangle \cdot x = \langle \tau_1 \cdots \tau_n \rangle \cdot x$; and claim follows. ∎

<u>Lemma</u> (Uniqueness) Suppose $\tau_1, \ldots, \tau_m$ are disjoint cycles and $\sigma_1, \ldots, \sigma_k$ are disjoint cycles. Suppose $|\operatorname{Supp} \tau_i| \geq 2$, $|\operatorname{Supp} \sigma_i| \geq 2$. Then $\tau_1 \cdots \tau_m = \sigma_1 \cdots \sigma_k$ implies $m = k$ and $\tau_1 = \sigma_{i_1}, \ldots, \tau_m = \sigma_{i_m}$ where $i_1, \ldots, i_m$ is a permutation of $[1..m]$.

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Pf. We proceed by induction on m; with an understanding that $m=0$ means the LHS is trivial.

Base of induction. If $k\neq 0$, $\operatorname{supp}(\sigma_1\cdots\sigma_k)=\bigcup_{i=1}^{k}\operatorname{supp}\sigma_i\neq\emptyset$ which is a contradiction.

Induction Step. Since τ_1 is a non-trivial cycle, $\operatorname{supp}\tau_1$ is an orbit of τ_1 that has at least 2 elements.

Hence $\operatorname{supp}\tau_1$ is an orbit of $\langle\tau_1\cdots\tau_m\rangle=\langle\sigma_1\cdots\sigma_k\rangle$.

So by the previous corollary, $\exists!\, i_1$ s.t. $\operatorname{supp}\tau_1$ is an orbit of σ_{i_1}. As σ_j's are cycles, $\operatorname{supp}\tau_1$ is the unique orbit of σ_{i_1} that has at least two elements. And so $\operatorname{supp}\sigma_{i_1}=\operatorname{supp}\tau_1$. Therefore

$$\begin{aligned}\langle\tau_1|_{\operatorname{supp}\tau_1}\rangle &= \langle(\tau_1\cdots\tau_m)|_{\operatorname{supp}\tau_1}\rangle\\ &= \langle(\sigma_1\cdots\sigma_k)|_{\operatorname{supp}\sigma_{i_1}}\rangle\\ &= \langle\sigma_{i_1}|_{\operatorname{supp}\sigma_{i_1}}\rangle;\ \text{which implies } \tau_1=\sigma_{i_1}.\end{aligned}$$

Now claim follows by commutativity of σ_j's and induction hypoth. ∎

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<u>Lemma</u> (Existence) For any $\sigma \in S_n \setminus \{I\}$, $\exists$ disjoint cycles τ_i's s.t. $\sigma = \tau_1 \cdots \tau_m$.

<u>Pf.</u> Suppose $\langle\sigma\rangle \backslash [1..n] = \{X_1, \dots, X_k\}$; and after reordering assume $|X_1|, \dots, |X_m| \geq 2$, and $|X_{m+1}| = \dots = |X_k| = 1$.

For $1 \leq i \leq m$, let $\tau_i \in S_n$, $\tau_i|_{X_i} := \sigma|_{X_i}$ and $\tau_i|_{X_i^c} = I|_{X_i^c}$

<u>Claim 1</u> τ_i is a cycle.

<u>Pf of Claim</u>. • $\tau_i(X_i) = \sigma(X_i) = X_i \Rightarrow \tau_i$ is surj $\Rightarrow \tau_i \in S_n$.

• $X_i = \langle\sigma\rangle \cdot x = \{x, \sigma(x), \dots, \sigma^{l-1}(x)\}$ and $\sigma^l(x) = x$

$= \{x, \tau_i(x), \dots, \tau_i^{l-1}(x)\}$

$\Rightarrow \tau_i = (x \;\; \sigma(x) \;\; \cdots \;\; \sigma^{l-1}(x))$. ∎

<u>Claim 2</u>. $\sigma = \tau_1 \cdots \tau_m$.

<u>Pf of Claim</u>. $\forall x \in \mathrm{Supp}\,\sigma$, $\exists!$ $i_x \in [1..m]$, $x \in X_{i_x}$.

$\Rightarrow \sigma(x) = \tau_{i_x}(x) = (\tau_1 \cdots \tau_m)(x)$

$$\boxed{(\tau_1 \cdots \tau_m)|_{\mathrm{Supp}\,\tau_{i_x}} = \tau_{i_x}|_{\mathrm{Supp}\,\tau_{i_x}}}$$

By Claim 1 and 2, $\tau_1 \cdots \tau_m$ is a cycle decomposition of σ. ∎

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<u>Proposition</u>. $\forall\ \sigma \in S_n \setminus \{I\}$ can be written as a product of disjoint cycles; and this decomposition is unique up to reordering its factors. (This decomposition is called the cycle decomposition of σ.)

<u>Lemma</u>. Suppose $\tau_1 \cdots \tau_m$ is a cycle decomposition of σ; and $|\mathrm{Supp}\, \tau_i| = l_i$. Then $o(\sigma) = \mathrm{l.c.m.}(l_1, \ldots, l_m)$.

<u>Pf.</u> Recall that disjoint permutations commute; and if $g_1 g_2 = g_2 g_1$, then $o(g_1 g_2) = \mathrm{l.c.m.}(o(g_1), o(g_2))$. Now prove the claim by induction on m. ∎

<u>Def</u> The <u>cycle type</u> of a permutation $\sigma \in S_n$ is the partition of n given by the size of orbits of $\langle \sigma \rangle$.

<u>Lemma</u>. If $\tau_1 \cdots \tau_m$ is a cycle decomposition of σ, then σ's cycle type is given by $|\mathrm{Supp}\, \tau_i|$'s and $\left(n - \sum_{i=1}^{m} |\mathrm{Supp}\, \tau_i|\right)$-many 1's.

<u>Pf</u> Exer. □

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<u>Cor.</u> Suppose $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_m$ is the cycle type of σ. Then

$o(\sigma) = \text{l.c.m.}(\ell_1, \ldots, \ell_m)$.

<u>Pf.</u> Suppose $\tau_1 \cdots \tau_k$ is a cycle decomposition of σ. Then

$\{|\text{Supp } \tau_i|\}_{i=1}^{k} \subseteq \{\ell_1, \ldots, \ell_m\} \subseteq \{|\text{Supp } \tau_i|\}_{i=1}^{k} \cup \{1\}$;

and so $\text{l.c.m}(\ell_1, \ldots, \ell_m) = \underset{1 \leq i \leq k}{\text{l.c.m.}}(|\text{Supp } \tau_i|)$; and claim follows. ■

<u>Lemma</u>. $\sigma(i_1 \cdots i_m)\sigma^{-1} = (\sigma(i_1) \cdots \sigma(i_m))$; and so if τ is an m-cycle, then $\sigma\tau\sigma^{-1}$ is an m-cycle and

$$\text{Supp}(\sigma\tau\sigma^{-1}) = \sigma(\text{Supp } \tau).$$

<u>Pf.</u> • $j \notin \{\sigma(i_1), \ldots, \sigma(i_m)\} \Leftrightarrow \sigma^{-1}(j) \notin \{i_1, \ldots, i_m\}$

$\Leftrightarrow \sigma^{-1}(j) \notin \text{Supp } \tau$ where $\tau = (i_1, \ldots, i_m)$

$\Leftrightarrow \tau(\sigma^{-1}(j)) = \sigma^{-1}(j)$

$\Leftrightarrow \sigma\tau\sigma^{-1}(j) = j$.

• $\sigma\tau\sigma^{-1}(\sigma(i_\ell)) = \sigma(\tau(i_\ell)) = \sigma(i_{\ell+1})$ where $i_{m+1} = i_1$. ■

Next we show that cycle type determines the conjugacy class.

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<u>Proposition</u> σ and σ' in S_n are conjugate if and only if they have the same cycle type.

<u>Pf.</u> $(\Rightarrow)$ Suppose $\tau_1 \cdots \tau_m$ is a cycle decomposition of σ.

Then $\gamma\sigma\gamma^{-1} = (\gamma\tau_1\gamma^{-1})(\gamma\tau_2\gamma^{-1})\cdots(\gamma\tau_m\gamma^{-1})$.

- Since τ_i is a cycle, $\gamma\tau_i\gamma^{-1}$ is a cycle.
- $\mathrm{Supp}(\gamma\tau_i\gamma^{-1}) = \gamma(\mathrm{Supp}\,\tau_i)$; in particular

$$(*) \qquad |\mathrm{Supp}(\gamma\tau_i\gamma^{-1})| = |\mathrm{supp}\,\tau_i|.$$

- $\mathrm{Supp}\,\tau_i \cap \mathrm{Supp}\,\tau_j = \emptyset \Rightarrow \gamma(\mathrm{Supp}\,\tau_i) \cap \gamma(\mathrm{Supp}\,\tau_j) = \emptyset$

$$\Rightarrow \mathrm{Supp}(\gamma\tau_i\gamma^{-1}) \cap \mathrm{supp}(\gamma\tau_j\gamma^{-1}) = \emptyset$$

So $(\gamma\tau_1\gamma^{-1})\cdots(\gamma\tau_m\gamma^{-1})$ is a cycle decomposition of $\gamma\sigma\gamma^{-1}$. Hence by $(*)$ claim follows.

$(\Leftarrow)$ Suppose $p_1 \le p_2 \le \cdots \le p_\ell$ is the cycle type of σ and σ'. So $\sigma = (\underbrace{\quad\quad}_{p_1})(\underbrace{\quad\quad}_{p_2})\cdots(\underbrace{\quad\quad}_{p_\ell})$ disjoint cycles. (possibily trivial)

and $\sigma' = (\underbrace{\quad\quad}_{p_1})(\underbrace{\quad\quad}_{p_2})\cdots(\underbrace{\quad\quad}_{p_\ell})$. So the blue arrows

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give us $\gamma \in S_n$ (as numbers written in paranth. are distinct).

And so $\gamma \sigma \gamma^{-1} = \sigma'$. ∎

Corollary. # of conjugacy classes of S_n =

# of partitions of n.

Remark. The above argument can help us enumerate $|Cl(\sigma)|$;

and so we can compute $C_{S_n}(\sigma)$.

Linking: $(a_0\, a_1 \cdots a_m)(a_m\, a_{m+1} \cdots a_\ell) = (a_0 \cdots a_\ell)$ if $a_i \neq a_j$.

Pf. It is enough to focus on a_i's. (The rest are fixed).

- $\forall\, 0 \leq i < m$, $(a_m \cdots a_\ell)\, a_i = a_i$

  $\Rightarrow (a_0\, a_1 \cdots a_m)(a_m \cdots a_\ell)\, a_i = a_{i+1}$.

- $m \leq i < \ell$, $(a_m \cdots a_\ell)\, a_i = a_{i+1}$ & $m < i+1 \leq \ell$, $(a_0 \cdots a_m)\, a_{i+1} = a_{i+1}$ $\Rightarrow (a_0 \cdots a_m)(a_m \cdots a_\ell)\, a_i = a_{i+1}$.

- $(a_0 \cdots a_m)(a_m \cdots a_\ell)\, a_\ell = (a_0 \cdots a_m)\, a_m = a_0$. ∎

And so $(a_0 \cdots a_m) = (a_0\, a_1)(a_1\, a_2) \cdots (a_{m-1}\, a_m)$ if $a_i \neq a_j$.

Def A 2-cycle is called a transposition.

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<u>Cor.</u> Any $\sigma \in S_n$ can be written as a product of transpositions.

<u>pf.</u> Any σ can be written as a product of cycles; and any cycle can be written as a product of transpositions. ∎

<u>Warning</u>. This product is NOT unique:

$$(1\ 2)(1\ 3)(1\ 2) = (2\ 3).$$
(conjugating $(1\ 3)$ by $(1\ 2)$)

We will show the parity of the number of transpositions is independent of the choice of such a product.

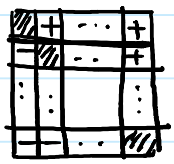
<u>Def.</u> $\Delta(x_1, \ldots, x_n) := \prod_{i<j} (x_i - x_j)$

- $\Delta_\sigma(x_1, \ldots, x_n) := \Delta(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

<u>Lemma</u>. $$\prod_{i \neq j} (x_i - x_j) = (-1)^{\frac{n(n-1)}{2}} \Delta^2$$
$$= (-1)^{\frac{n(n-1)}{2}} \Delta_\sigma^2;$$

in particular $\exists\ \epsilon(\sigma) \in \{\pm 1\}$ s.t. $\Delta_\sigma = \epsilon(\sigma)\Delta$.

<u>Pf.</u> $\cdot\ (x_i - x_j)(x_j - x_i) = -(x_i - x_j)^2$



. And we get $\frac{n(n-1)}{2}$-many (-1) factors.

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And so $\prod_{i\neq j}(x_i-x_j)=(-1)^{\frac{n(n-1)}{2}}\Delta^2$.

Hence

$$(-1)^{\frac{n(n-1)}{2}}\Delta_\sigma^2=(-1)^{\frac{n(n-1)}{2}}\Delta(x_{\sigma(1)},\dots,x_{\sigma(n)})^2$$

$$=\prod_{i\neq j}(x_{\sigma(i)}-x_{\sigma(j)})$$

$$=\prod_{i\neq j}(x_i-x_j).$$

Therefore $\Delta^2=\Delta_\sigma^2$, which implies $\Delta_\sigma=\epsilon(\sigma)\,\Delta$ for some $\epsilon(\sigma)\in\{\pm 1\}$. ∎