At the end of the previous lecture we stated the following lemma:

**Lemma.** Suppose \( \tau_i \in S_m, \supp \tau_i \cap \supp \tau_j = \emptyset \) if \( i \neq j \).

Then \( \left( \tau_1 \ldots \tau_n \right) \mid_{\supp \tau_i} = \tau_i \mid_{\supp \tau_i} \); and

\[ \supp \left( \tau_1 \ldots \tau_n \right) = \bigcup_{i=1}^{n} \supp \tau_i. \]

In particular \( \left( \tau_1 \ldots \tau_n \right) \mid_{\supp \tau_i} \in S_{\supp \tau_i} \).

**Proof.** \( \supp \tau_i \subseteq \bigcup_{j \neq i} \Fix \tau_j \) as \( \tau_j \)'s are disjoint.

Since \( \tau_i(\supp \tau_i) = \supp \tau_i \), \( \forall x \in \supp \tau_i \)

\[ \tau_i(\tau_{i+1} \ldots \tau_n)(x) = \tau_i(x) \quad \text{and} \quad \tau_1 \ldots \tau_{i-1}(\tau_i(x)) = \tau_i(x). \]

And so \( \tau_1 \ldots \tau_n(x) = \tau_i(x) \). Therefore \( \supp \tau_i \subseteq \supp \tau_1 \ldots \tau_n \).

Thus \( \supp \tau_1 \ldots \tau_n = \bigcup_{i=1}^{n} \supp \tau_i \).

We also deduce \( \left( \tau_1 \ldots \tau_n \right)(\supp \tau_i) = \tau_i(\supp \tau_i) \)

\[ = \supp \tau_i; \]

and so \( \left( \tau_1 \ldots \tau_n \right) \mid_{\supp \tau_i} \in S_{\supp \tau_i} \).

**Corollary.** Suppose \( \supp \tau_i \cap \supp \tau_j = \emptyset \) if \( i \neq j \), \( X \subseteq [1 \ldots n] \), and \( |X| \geq 2 \). Then \( X \) is an orbit of \( \left< \tau_1 \ldots \tau_n \right> \) if and only if
\( X \) is an orbit of \( \tau_i \) for some \( i \).

\[ \iff \]

**Proof:** Suppose \( X \) is the orbit of \( x \in X \). Since \( |X| \geq 2 \), \( x \in \text{Supp } \tau_1 \cdots \tau_n = \bigcup_{i=1}^{n} \text{Supp } \tau_i \). So \( \exists i \) s.t. \( x \in \text{Supp } \tau_i \).

(by the previous Lemma)

Again by the previous lemma \( \langle \tau_1 \cdots \tau_n \rangle = \langle \tau_i \rangle_{\text{Supp } \tau_i} \).

Hence \( \langle \tau_1 \cdots \tau_n \rangle \cdot x = \langle \tau_i \rangle \cdot x \); and claim follows.

\[ \iff \]

Suppose \( X = \langle \tau_i \rangle \cdot x \). Then \( x \in \text{Supp } \tau_i \); again by the previous lemma \( \langle \tau_1 \cdots \tau_n \rangle_{\text{Supp } \tau_i} = \langle \tau_i \rangle_{\text{Supp } \tau_i} \); and so \( \langle \tau_i \rangle \cdot x = \langle \tau_1 \cdots \tau_n \rangle \cdot x \); and claim follows.

Lemma (Uniqueness) Suppose \( \tau_1, \ldots, \tau_m \) are disjoint cycles and \( \sigma_1, \ldots, \sigma_k \) are disjoint cycles. Suppose \( |\text{Supp } \tau_i| \geq 2 \), \( |\text{Supp } \sigma_i| \geq 2 \). Then \( \tau_1 \cdots \tau_m = \sigma_1 \cdots \sigma_k \) implies \( m = k \) and \( \tau_i = \sigma_{i_1} \), \ldots, \( \tau_m = \sigma_{i_m} \) where \( i_1, \ldots, i_m \) is a permutation of \( [1 \ldots m] \).
Proof. We proceed by induction on $m$; with an understanding that $m=0$ means the LHS is trivial.

**Base of induction.** If $k \neq 0$, \( \text{Supp} (\sigma_1 \ldots \sigma_k) = \bigcup_{i=1}^{k} \text{Supp} \sigma_i \neq \emptyset \)
which is a contradiction.

**Induction step.** Since \( \tau_1 \) is a non-trivial cycle, \( \text{Supp} \tau_1 \) is an orbit of \( \tau_1 \) that has at least 2 elements. Hence \( \text{Supp} \tau_1 \) is an orbit of \( \langle \tau_1 \ldots \tau_m \rangle = \langle \sigma_1 \ldots \sigma_k \rangle \).

So by the previous corollary, \( \exists \ i_1 \text{ s.t. } \text{Supp} \tau_1 \) is an orbit of \( \sigma_{i_1} \). As \( \sigma_0 \)'s are cycles, \( \text{Supp} \tau_1 \) is the unique orbit of \( \sigma_{i_1} \) that has at least two elements. And so \( \text{Supp} \sigma_{i_1} = \text{Supp} \tau_1 \). Therefore

\[
\langle \tau_1 \mid_{\text{Supp} \tau_1} \rangle = \langle \tau_1 \ldots \tau_m \rangle \mid_{\text{Supp} \tau_1} = \langle \sigma_1 \ldots \sigma_k \rangle \mid_{\text{Supp} \sigma_{i_1}} = \langle \sigma_{i_1} \mid_{\text{Supp} \sigma_{i_1}} \rangle ; \text{ which implies } \tau_1 = \sigma_{i_1}.
\]

This claim follows by commutativity of \( \sigma_0 \)'s and induction hypothesis. \( \blacksquare \)
Lemma (Existence) For any $\sigma \in S_n \backslash \mathcal{I}$, $\exists$ disjoint cycles $\tau_i$'s s.t. $\sigma = \tau_1 \ldots \tau_m$.

Proof. Suppose $\langle 1 \ldots n \rangle = \mathcal{I}_1, \ldots, \mathcal{I}_k$; and after reordering assume $|X_1|, \ldots, |X_m| \geq 2$, and $|X_{m+1}| = \ldots = |X_k| = 1$.

For $1 \leq i \leq m$, let $\tau_i \in S_n$, $\tau_i|_{X_i} = \sigma|_{X_i}$ and $\tau_i|_{X_i^c} = I|_{X_i^c}$.

Claim 1. $\tau_i$ is a cycle.

Proof of Claim. $\tau_i(X_i) = \sigma(X_i) = X_i \Rightarrow \tau_i$ is surj $\Rightarrow \tau_i \in S_n$.

$\Rightarrow \tau_i = (x \sigma(x) \ldots \sigma^{t_i-1}(x))$. $\quad \Box$

Claim 2. $\sigma = \tau_1 \ldots \tau_m$.

Proof of Claim. $\forall x \in \text{Supp } \sigma, \exists! \ i_x \in \{1 \ldots m\}, x \in X_{i_x}$.

$\Rightarrow \sigma(x) = \tau_{i_x}(x) = (\tau_1 \ldots \tau_m)(x)$.

By Claim 1 and 2, $\tau_1 \ldots \tau_m$ is a cycle decomposition of $\sigma$. $\blacksquare$
Proposition. A $\sigma \in S_n \setminus \{\emptyset\}$ can be written as a product of disjunct cycles; and this decomposition is unique up to reordering its factors. (This decomposition is called the cycle decomposition of $\sigma$.)

Lemma. Suppose $\tau_1 \ldots \tau_m$ is a cycle decomposition of $\sigma$; and $|\text{Supp} \tau_i| = l_i$. Then $o(\sigma) = \text{l.c.m.}(l_1, \ldots, l_m)$.

Proof. Recall that disjunct permutations commute, and if $g_1 g_2 = g_2 g_1$, then $o(g_1 g_2) = \text{l.c.m.}(o(g_1), o(g_2))$. Now prove the claim by induction on $m$.

Definition. The cycle type of a permutation $\sigma \in S_n$ is the partition of $n$ given by the size of orbits of $\langle \sigma \rangle$.

Lemma. If $\tau_1 \ldots \tau_m$ is a cycle decomposition of $\sigma$, then $\sigma$'s cycle type is given by $|\text{Supp} \tau_i|$'s and $(n - \sum_{i=1}^{m} |\text{Supp} \tau_i|)$ - many 1's.

Exer. □
Con. Suppose \( l_1 \leq l_2 \leq \ldots \leq l_m \) is the cycle type of \( \sigma \). Then
\[
\sigma(\sigma') = \text{l.c.m.}(l_1, \ldots, l_m).
\]

Pf. Suppose \( \tau_1, \ldots, \tau_k \) is a cycle decomposition of \( \sigma \). Then
\[
\forall_{i=1}^k \text{Supp } \tau_i \mid \sum_{i=1}^k \text{Supp } \tau_i \leq \sum_{i=1}^k \text{Supp } \tau_i \cup \frac{1}{l_i} f_i.
\]
and so \( \text{l.c.m.}(l_1, \ldots, l_m) = \text{l.c.m.}(\text{Supp } \tau_i) \); and claim follows.

\[\text{Lemma. } \sigma(i_1, \ldots, i_m) \sigma^{-1} = (\sigma(i_1) \ldots \sigma(i_m)) ; \text{ and so if } \tau \text{ is an } m\text{-cycle, then } \sigma \tau \sigma^{-1} \text{ is an } m\text{-cycle and}
\]
\[
\text{Supp}(\sigma \tau \sigma^{-1}) = \sigma(\text{Supp } \tau).
\]

Pf. \( i_j \notin \sigma(i_1), \ldots, \sigma(i_m) \iff \sigma^{-1}(j) \notin \frac{1}{l_1} i_1, \ldots, \frac{1}{l_m} i_m \)
\[
\iff \sigma^{-1}(j) \notin \text{Supp } \tau \text{ where}
\]
\[
\tau = (i_1, \ldots, i_m)
\]
\[
\iff \tau(\sigma^{-1}(j)) = \sigma^{-1}(j)
\]
\[
\iff \sigma \tau \sigma^{-1}(j) = j.
\]
\[
\sigma \tau \sigma^{-1}(\sigma(i_1)) = \sigma(\tau(i_2)) = \sigma(i_{l+1}) \text{ where } \frac{i_{l+1}}{l_{l+1}} = i_{l+1}.
\]

Next we show that cycle type determines the conjugacy class.
Proposition $\sigma$ and $\sigma'$ in $S_n$ are conjugate if and only if they have the same cycle type.

Proof ($\iff$) Suppose $\tau_1, \ldots, \tau_m$ is a cycle decomposition of $\sigma$.

Then $\forall \tau \forall \tau'^{-1} = (\forall \tau_1 \forall^{-1})(\forall \tau_2 \forall'^{-1}) \ldots (\forall \tau_m \forall'^{-1})$.

Since $\tau_i$ is a cycle, $\forall \tau_i \forall^{-1}$ is a cycle.

$\supp(\forall \tau_i \forall^{-1}) = \forall (\supp \tau_i)$; in particular

$|\supp(\forall \tau_i \forall^{-1})| = |\supp \tau_i|.

\supp \tau_i \cap \supp \tau_j = \emptyset \Rightarrow \forall (\supp \tau_i) \cap \forall (\supp \tau_j) = \emptyset

\Rightarrow \supp(\forall \tau_i \forall^{-1}) \cap \supp(\forall \tau_j \forall^{-1}) = \emptyset

\forall (\forall \tau_1 \forall^{-1}) \ldots (\forall \tau_m \forall^{-1})$ is a cycle decomposition of $\forall \sigma \forall^{-1}$. Hence by $\forall \sigma$ claim follows.

$\iff$ Suppose $p_1 \leq p_2 \leq \ldots \leq p_d$ is the cycle type of $\sigma$ and $\sigma'$. So $\sigma = (\Diagram{p_1}) (\Diagram{p_2}) \ldots (\Diagram{p_d})$ disjoint cycles.

and $\sigma' = (\Diagram{p_1}) (\Diagram{p_2}) \ldots (\Diagram{p_d})$. So the blue arrows
Lecture 08: Linking and transpositions

Tuesday, October 23, 2018  5:39 PM

give us $\gamma \in S_n$ (as numbers written in parentheses are distinct).

And so $\gamma \circ \gamma^{-1} = \sigma'$.  

Corollary. # of conjugacy classes of $S_n = \#	ext{ of partitions of } n$.

Remark. The above argument can help us enumerate $|Cl(\sigma)|$;
and so we can compute $C_{S_n}(\sigma')$.

Linking : $(a_0 a_1 \ldots a_m) (a_m a_m+1 \ldots a_l) = (a_0 \ldots a_l)$ if $a_i \neq a_j$.

Proof. It is enough to focus on $a_i$'s. (The rest are fixed).

$$\forall 0 \leq i < m, \quad (a_m \ldots a_l) \cdot a_i = a_i$$

$$\Rightarrow (a_0 a_1 \ldots a_m) (a_m \ldots a_l) \cdot a_i = a_{i+1}.$$  

$m \leq i < l$, $(a_m \ldots a_l) \cdot a_i = a_{i+1} \Rightarrow (a_0 \ldots a_m) (a_m \ldots a_l) \cdot a_i = a_{i+1}$.

$m < i+1 \leq l$, $(a_0 \ldots a_m) \cdot a_{i+1} = a_{i+1}.

(a_0 \ldots a_m) (a_m \ldots a_l) \cdot a_l = (a_0 \ldots a_m) a_m = a_o$.  

And so $(a_0 \ldots a_m) = (a_0 a_1) (a_1 a_2) \ldots (a_{m-1} a_m)$ if $a_i \neq a_j$.

Def. A 2-cycle is called a transposition.
Cor. Any $\sigma \in S_n$ can be written as a product of transpositions.

Proof. Any $\sigma$ can be written as a product of cycles; and any cycle can be written as a product of transpositions.

Warning. This product is not unique:

$$ (1 \ 2) (1 \ 3) (1 \ 2) = (2 \ 3) \cdot$$

(conjugating $(1 \ 3)$ by $(1 \ 2)$)

We will show the parity of the number of transpositions is independent of the choice of such a product.

**Def.** $\Delta (x_1, \ldots, x_n) := \prod_{i<j} (x_i - x_j)$

- $\Delta_\sigma (x_1, \ldots, x_n) := \Delta (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

**Lemma.** $\prod_{i \neq j} (x_i - x_j) = (-1)^{\frac{n(n-1)}{2}} \Delta^2$

$$= (-1)^{\frac{n(n-1)}{2}} \Delta^2_\sigma;$$

In particular $\exists \in (\sigma) \in \mathfrak{S}_{\leq n}$ s.t. $\Delta_\sigma = \epsilon(\sigma) \Delta$.

**Proof.** $(x_i - x_j)(x_j - x_i) = -(x_i - x_j)^2$ and we get $\frac{n(n-1)}{2}$ many $(-1)$ factors.
And so \[ \prod_{1 \neq j} (x_i - x_j) = (1)^{\frac{n(n-1)}{2}} \Delta^2. \]

Hence \[ (1)^{\frac{n(n-1)}{2}} \Delta_{\sigma} = (1)^{\frac{n(n-1)}{2}} \Delta(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \]

\[ = \prod_{1 \neq j} (x_{\sigma(i)} - x_{\sigma(j)}) \]

\[ = \prod_{1 \neq j} (x_i - x_j). \]

Therefore \[ \Delta^2 = \Delta_{\sigma}^2, \] which implies \[ \Delta_{\sigma} = \epsilon(\sigma) \Delta \] for some \[ \epsilon(\sigma) \in \mathbb{S}_n^\pm 1. \]