Def: A group $G$ is called **solvable** if \( \exists k \in \mathbb{Z}^{>0}, \ G^{(k)} = 1. \)

(Name is given because of a theorem by Galois on solvability of a polynomial by radicals.)

**Lemma.** $\phi: G \to H$ a group homomorphism \( \Rightarrow \)

\[ \forall i, \ \phi(G^{(i)}) = \phi(G)^{(i)}. \]

**Prop.** Exercise, prove this by induction on $i$.

**Theorem.** Suppose $G$ is a solvable group, $H \leq G$, and $N \leq G$.

Then (1) $H$ is solvable, (2) $G/N$ is solvable.

**Pf.** (1) By induction on $i$, \( H^{(i)} \subseteq G^{(i)} \) \( \Rightarrow \)

\[ H^{(k)} = 1 \]

\[ G^{(k)} = 1 \]

(2) $\pi: G \to G/N$ \( \Rightarrow \)

\[ \pi(G)^{(k)} = \pi(G^{(k)}) = 1 \Rightarrow (G/N)^{(k)} = 1. \]

**Proposition.** Suppose $G$ is a finite group. Then

$G$ is solvable \( \iff \) all the composition factors of $G$

are cyclic groups of prime order.

**Pf.** ($\Rightarrow$) Suppose \( 1 = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_k = G \) is a composition series of $G$. Then $N_i$'s are solvable, and so $N_i/N_{i-1}$'s are solvable. Therefore the following claim implies this direc.
Claim. A solvable simple group is a cyclic group of prime order. 

\[ \begin{align*}
\text{If of Claim. } & \quad H^{(k)} = 1 \iff H^{(d)} \neq H \iff H^{(d)} = 1 \Rightarrow H \text{ abelian} \\
& \Rightarrow \text{ any subgroup is normal } \iff H \text{ is a cyclic group of prime order. } \\
& \quad \square
\end{align*} \]

(\Leftarrow) Suppose \(1 = N_0 < \cdots < N_k = G\) is a composition series and \(N_i/N_{i+1}\) is cyclic. Then by a lemma that we proved earlier \(G^{(k)} = 1\) and so \(G\) is solvable. 

Def. Let \( \gamma_1(G) := G, \gamma_{i+1}(G) := [\gamma_i(G), G] \). \( \bigcap_{i=1}^{\infty} \gamma_i(G) \) is called the lower central series of \(G\). 

- Let \( Z_0(G) := \{1\}, Z(G/Z_i(G)) = Z_{i+1}(G)/Z_i(G) \). \( \bigcap_{i=0}^{\infty} Z_i(G) \) is called the upper central series of \(G\). 

Basic Properties 
(1) \( \gamma_i(G) \) is a char. subgroup 
(2) \( \gamma_1(G) \supseteq \gamma_2(G) \supseteq \cdots \) 
(3) \( Z(G/\gamma_{i+1}(G)) \supseteq \gamma_i(G)/\gamma_{i+1}(G) \). 


(4) \( Z_i(C_G) \) is a char. subgp.

(5) \( Z_{i+1}(G)/Z_i(C_G) \) is abelian.

**Pf.** (1) We proceed by induction on \( i \); base case is clear.

\[
\forall \theta \in \text{Aut}(G), \quad \theta(\gamma_{i+1}(G)) = \theta([\gamma_i(G), G]) = [\theta(\gamma_i(G)), \theta(G)]
\]

\[= [\gamma_i(G), G] = \gamma_{i+1}(G).\]

(2) \([\gamma_i(C_G), G] \subseteq \gamma_i(C_G) \) as \( \gamma_i(C_G) \triangleleft G \).

(3) \( \forall g \in G, g' \in \gamma_i(C_G), \ [g, g'] \in \gamma_{i+1}(G) \); and so

\[
(\gamma_{i+1}(G))(g' \gamma_{i+1}(G)) = (g' \gamma_{i+1}(G))(g \gamma_{i+1}(G)), \text{ which}
\]

implies \( g' \gamma_{i+1}(G) \in Z(G/\gamma_{i+1}(G)) \).

(4) We proceed by induction on \( i \); the base case is clear.

\( \forall \theta \in \text{Aut}(G), \) since \( \Theta(Z_i(G)) = Z_i(G) \),

\[
\overline{\Theta}: G/Z_i(C_G) \to G/Z_i(C_G), \quad \overline{\Theta}(gZ_i(G)) := \Theta(g)Z_i(G) \text{ is}
\]
a well-defined automorphism; and so \( \overline{\Theta}(Z(G/Z_i(G))) = Z(G/Z_i(G)) \).

This implies \( \Theta(Z_{i+1}(G)) Z_i(C_G) = Z_{i+1}(G) \Rightarrow \Theta(Z_{i+1}(G)) = Z_{i+1}(G) \)

\( \overline{\Theta}(Z_i(C_G)) \)
(5) \( Z_{i+1}(G_i)/Z_i(G_i) = Z(G_i/Z_i(G_i)) \); and so it is abelian.

**Theorem**: For a non-negative integer \( c \),

\[
\gamma_{c+1}(G) = \gamma_{c+1} \iff Z_c(G) = G.
\]

**Pf.** \((\Rightarrow)\) We prove by induction on \( i \) that

\[
\gamma_{c+1-i}(G) \subseteq Z_i(G).
\]

**Base** follows from \( \gamma_{c+1}(G) = \gamma_{c+1} \).

**Induction Step.** To show \( \gamma_{c-i}(G) \subseteq Z_{i+1}(G) \), one has to show \( \forall g' \in \gamma_{c-i}(G), \ g' Z_i(G) \subseteq Z(G_i/Z_i(G_i)) \); that means

\[
\forall g \in G, \ [g, g'] \text{ should be in } Z_i(G).
\]

\[
[g, g'] \in [G, \gamma_{c-i}(G)] = \gamma_{c-i+1}(G) \subseteq Z_i(G), \text{ and claim follows. And so } G = \gamma_1(G) \subseteq Z_c(G).
\]

\((\Leftarrow)\) By induction on \( i \), we prove \( \gamma_i(G) \subseteq Z_{c+i-1}(G) \).

Since \( Z_c(G) = G \), the base case follows.

To prove the induction step, we have to show \( \gamma_{i+1}(G_i) \subseteq Z_{c-i}(G) \).
Lecture 11: Nilpotent groups

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Notice that \( Z_{c-i+1}(G)/Z_{c-i}(G) = Z(G/Z_{c-i}(G)) \).

\[
\Rightarrow [G, Z_{c-i+1}(G)] \subseteq Z_{c-i}(G) \quad \Rightarrow [G, Z_{c-i}(G)] \subseteq Z_{c-i}(G)
\]

\[
\Rightarrow \gamma_i(G) \subseteq Z_{c-i}(G) \quad \Rightarrow \gamma_{i+1}(G) \subseteq Z_{c-i}(G);
\]

and claim follows. In particular, \( \gamma_{c+1}(G) \subseteq Z_c(G) = \mathbb{Z}_c \).

**Def.** A group \( G \) is called **nilpotent** if \( \exists c \in \mathbb{Z}^+ \) s.t.

\[
\gamma_{c+1}(G) = 1.
\]

(Alternatively \( Z_c(G) = G \)). In this case we say the

nilpotency class of \( G \) is \( c \).

**Proposition.** A finite \( p \)-group is nilpotent.

**Pr.** We proceed by strong induction on \( |G| \). If \( |G| = 1 \), we are done. If not, \( Z(G) \neq 1 \). And so \( |G/Z(G)| < |G| \) and \( G/Z(G) \) is a finite \( p \)-group. So \( \exists c \in \mathbb{Z}^+ \) s.t. the \( c \)-th group in the upper central series of \( G/Z(G) \) is \( G/Z(G) \).

Notice that the upper central series of \( G/Z(G) \) is \( \{Z_i(G)/Z(G)\}_{i=1}^\infty \); and so \( Z_c(G) = G \).
Proposition. Suppose $G$ is a nilpotent group. Then

$$H \leq G \Rightarrow H \leq N_G(H).$$

Pf. Since $G$ is nilpotent, $\exists c \in \mathbb{Z}^+, Z_c(G) = G$. Let $i_0 < c$ be s.t. $Z_{i_0}(G) \subseteq H$ and $Z_{i_0+1}(G) \not\subseteq H$.

Let $g \in Z_{i_0+1}(G) \setminus H \Rightarrow g Z_{i_0}(G) \in Z(G/Z_{i_0}(G))$

$$g \not\in H$$

$$\Rightarrow [g, H] \subseteq Z_{i_0}(G) \subseteq H \Rightarrow \forall h \in H, g^{-1} h^{-1} g h \in H$$

$$\Rightarrow g^{-1} h^{-1} g \in H \Rightarrow g^{-1} H g \subseteq H$$

Similarly $g H g^{-1} \subseteq H \Rightarrow g \in N_G(H) \setminus H$.

Corollary. Suppose $G$ is a finite nilpotent group. Then any $Sylow$ $p$-subgp of $G$ is normal.

Pf. We know $N_G(N_G(P)) = N_G(P)$ if $P$ is a $Sylow$ $p$-subgp.

Hence, by the above proposition, $N_G(P) = G$; this means $P \triangleleft G$. $\blacksquare$
Theorem. Suppose $G$ is a finite group. Then the following statements are equivalent:

1. $G$ is nilpotent.
2. $\forall p \mid |G|$, $G$ has a unique Sylow $p$-subgroup.
3. $G \cong \prod_{i=1}^{m} P_i$ where $P_i$ is a finite $p_i$-group.

Proof. (1) $\Rightarrow$ (2) is proved in the previous corollary.

(2) $\Rightarrow$ (3) Suppose $|G| = \prod_{i=1}^{m} p_i^{k_i}$ where $k_i \in \mathbb{Z}^+$ and $p_i$'s are distinct prime numbers. Let $P_i$ be the unique Sylow $p_i$-subgroup of $G$. So $P_i \triangleleft G$; and since $\gcd(1, p_i, p_j) = 1$ for $i \neq j$, $P_i$ and $P_j$ commute. In particular, for any $l$,

\[ P_1 P_2 \cdots P_l \text{ is a normal subgroup of } G. \]

Claim. $P_1 \times P_2 \times \cdots \times P_l \xrightarrow{\phi} P_1 P_2 \cdots P_l$ is an isomorphism.

Proof of Claim. We proceed by induction $l$. The base case is clear.
Lecture 11: Characterization of finite nilpotent groups

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Induction Step: \[ P_1 \times \cdots \times P_\ell \times P_{\ell+1} \cong P_1 \cdot P_2 \cdot \cdots \cdot P_\ell \times P_{\ell+1} \]

By the induction hypothesis, \((\phi_{i}, \text{id.})\) is an isomorphism (and \(P_1 \cdot \cdots \cdot P_\ell \cdot P_{\ell+1} = \prod_{i=1}^{\ell+1} P_i\)). So it is enough to show the 2nd map is a group homomorphism.

\[ P_1 \cdot \cdots \cdot P_\ell \triangleleft G \]

\[ P_{\ell+1} \triangleleft G \]

\[ g \text{ and } \prod_{i=1}^{\ell} P_i \text{ commute.} \]

Based on these it is easy to see why \(\phi\) is an isomorphism (why?)

\((3) \Rightarrow (1)\) We have already proved that \(P_i\)’s are nilpotent. So \(\exists c \in \mathbb{Z}^+\) st. \(\gamma_c(P_i) = \mathbb{Z}^c\) for \(1 \leq i \leq m\). Then

\[ \gamma_c(G) = \prod_{i=1}^{m} \gamma_c(P_i) = \mathbb{Z}^c, \]

(Justify this)

Let me finish today’s lecture by another characterization of finite nilpotent groups:

**Theorem.** A finite group is nilpotent if and only if all maximal
subgroups are normal.

\[
\text{Pf. } (\Rightarrow) \quad M \neq G \implies M \leq N_G(M) \neq G \implies N_G(M) = G \implies M \vartriangleleft G.
\]

\[
\text{M is maximal}
\]

\[
(\Leftarrow) \quad \text{Let } P \text{ be a Sylow } p\text{-subgp. We would like to show } P \triangleleft G. \text{ Suppose to the contrary } N_G(P) \neq G. \text{ Then there is a maximal subgroup } M \text{ of } G \text{ st. } N_G(P) \leq M \text{ (since } G \text{ is finite, there is such a subgroup.) And so } M \vartriangleleft G \text{ and } P \text{ is a Sylow } p\text{-subgroup of } M. \text{ Hence by Fratini's argument } G = N_G(P). M \leq M \text{ which is a contradiction.} \]