Lecture 13: Universal property of free product

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In the previous lecture we defined the free product of groups $\{G_i\}_{i\in I}$:

$X := \bigsqcup_{i\in I} G_i$, $\mathcal{F}(X) := \mathcal{L}(X)/\sim$. We discussed why $\mathcal{F}(X)$ is a group. We denote it by $\mathop{*}_{i\in I} G_i$.

The universal property of free product of groups.

(Warning. In category theory, this is called the coproduct of these objects.)

Suppose G is a group and $f_i : G_i \to G$ are group homomorphisms. Then there is a unique group homomorphism $\tilde{f} : \mathop{*}_{i\in I} G_i \to G$ such that $\tilde{f}|_{G_i} = f_i$. Alternatively

$$\mathrm{Hom}\left(\mathop{*}_{i\in I} G_i, G\right) \longrightarrow \prod_{i\in I} \mathrm{Hom}(G_i, G)$$
$$\tilde{f} \longmapsto (\tilde{f}|_{G_i})_{i\in I}$$

is a bijection.

<u>Pf.</u> Let X be the disjoint union of G_i's; and $\mathcal{L}(X)$ be the free monoid generated by X. Let $f : X \to G$,
$$f(x) := f_i(x) \text{ if } x \in G_i.$$

Since $\mathcal{L}(X)$ is the free monoid generated by X, there is a monoid homomorphism $\hat{f} : \mathcal{L}(X) \to G$ such that $\hat{f}|_X = f$. That means

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$\hat{f}|_{G_i} = f_i$ is a group homomorphism; and so

- $\hat{f}(e_{G_i}) = e_G$ where e_{G_i} is the neutral element of G_i and e_G is the neutral element of G.
- $\hat{f}(x_3) = \hat{f}(x_1)\,\hat{f}(x_2)$ if $x_1, x_2 \in G_i$ and $x_3 = x_1 \cdot x_2$.

Next we show $\omega_1 \sim \omega_2 \Rightarrow \hat{f}(\omega_1) = \hat{f}(\omega_2)$.

Since $\sim$ is generated by the following relations, $\omega_1 e_{G_i} \omega_2 \sim \omega_1 \omega_2$

and $\omega_1 x_1 x_2 \omega_2 \sim \omega_1 x_3 \omega_2$ if $x_1, x_2 \in G_i$ and $x_3 = x_1 \cdot x_2$, it is

enough to show

$$\hat{f}(\omega_1 e_{G_i} \omega_2) = \hat{f}(\omega_1 \omega_2) \quad (1) \qquad \text{and} \qquad \hat{f}(\omega_1 x_1 x_2 \omega_2) = \hat{f}(\omega_1 x_3 \omega_2) \quad (2)$$

if $x_1, x_2 \in G_i$ and $x_3 = x_1 \cdot x_2$.

(1)
$$\begin{aligned}
\hat{f}(\omega_1 e_{G_i} \omega_2) &= \hat{f}(\omega_1)\,\hat{f}(e_{G_i})\,\hat{f}(\omega_2) && \hat{f} \text{ is a monoid homomorphism} \\
&= \hat{f}(\omega_1)\, e_G\, \hat{f}(\omega_2) && \hat{f}|_{G_i} \text{ is a gp hom.} \\
&= \hat{f}(\omega_1)\,\hat{f}(\omega_2) \\
&= \hat{f}(\omega_1 \omega_2) && \hat{f} \text{ is a monoid homo.}
\end{aligned}$$

(2)
$$\begin{aligned}
\hat{f}(\omega_1 x_1 x_2 \omega_2) &= \hat{f}(\omega_1)\,\hat{f}(x_1)\,\hat{f}(x_2)\,\hat{f}(\omega_2) && \text{monoid hom.} \\
&= \hat{f}(\omega_1)\,\hat{f}(x_3)\,\hat{f}(\omega_2) && \hat{f}|_{G_i} \in \mathrm{Hom}(G_i, G)
\end{aligned}$$

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$$= \hat{f}(\omega_1 x_3 \omega_2) \qquad \text{monoid hom.}$$

Let $\tilde{f}([\omega]) := \hat{f}(\omega)$. The previous claim shows that $\tilde{f}$ is well-defined.

<u>Claim.</u> $\tilde{f} \in \mathrm{Hom}(\ast_{i\in I} G_i, G)$.

<u>Pf.</u> $\tilde{f}([\omega_1][\omega_2]) = \tilde{f}([\omega_1\omega_2]) = \hat{f}(\omega_1\omega_2) = \hat{f}(\omega_1)\hat{f}(\omega_2)$

$$= \tilde{f}([\omega_1])\tilde{f}([\omega_2])$$

. $\tilde{f}([\emptyset]) = \hat{f}(\emptyset) = e_G$

. $\tilde{f}([x_1 \cdots x_n]^{-1}) = \tilde{f}([x_n^{-1} \cdots x_1^{-1}]) = \hat{f}(x_n^{-1} \cdots x_1^{-1})$

$$= \hat{f}(x_n^{-1})\hat{f}(x_{n-1}^{-1}) \cdots \hat{f}(x_1^{-1}) \qquad \text{monoid hom.}$$

$$= \hat{f}(x_n)^{-1}\hat{f}(x_{n-1})^{-1} \cdots \hat{f}(x_1)^{-1} \qquad \hat{f}|_{G_i} \in \mathrm{Hom}(G_i, G)$$

$$= (\hat{f}(x_1)\hat{f}(x_2) \cdots \hat{f}(x_n))^{-1}$$

$$= \hat{f}(x_1 x_2 \cdots x_n)^{-1}.$$

$$= \tilde{f}([x_1 \cdots x_n])^{-1}. \qquad \blacksquare$$

<u>Claim.</u> $\tilde{f}|_{G_i} = f_i$.

<u>Pf.</u> $\forall\, x \in G_i$, $\tilde{f}([x]) = \hat{f}(x) = f_i(x)$. $\blacksquare$

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We have proved the existence of $\tilde{f} \in \mathrm{Hom}(\ast_{i\in I} G_i, G)$ s.t. $\tilde{f}|_{G_i} = f_i$. The uniqueness is clear as $\ast_{i\in I} G_i$ is generated by $X = \bigcup_{i\in I} G_i$. ∎

<u>Ex.</u> Suppose a group G is generated by two elements a and b of order 2. Then $\exists\ (\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/2\mathbb{Z}) \overset{\phi}{\twoheadrightarrow} G$ which is an onto group homomorphism.

<u>Pf.</u> Let $\phi_a : \mathbb{Z}/2\mathbb{Z} \to G$, $1 \mapsto a$, $\phi_b : \mathbb{Z}/2\mathbb{Z} \xrightarrow{\sim} G$, $1 \mapsto b$. Then by the universal property of free products $\exists\ \phi : \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z} \to G$ s.t. ϕ restricted to $\mathbb{Z}/2\mathbb{Z}$'s gives us ϕ_a and ϕ_b. In parti. $a, b \in \mathrm{Im}\,\phi$; and so ϕ is onto. ∎

<u>Remark</u>. Later you will show $\mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}$ is solvable; and using the above example you can deduce $\langle a, b\rangle$ is solvable if $a^2 = b^2 = 1$.

<u>Def.</u> For any non-empty set X, the free group generated by X is denoted by $F(X)$ and it is the free product of

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$|X|$ copies of $\mathbb{Z}$; $F(X) = \underset{x\in X}{*} \mathbb{Z}$. To make these $\mathbb{Z}$ groups related to X, we write $F(X) = \underset{x\in X}{*} \langle x\rangle$ where $\langle x\rangle = \{x^n \mid n\in\mathbb{Z}\}$.

Universal Property of free groups

Suppose $X\neq\emptyset$, G is a group, and $f: X\to G$ is a function. Then $\exists!\ \hat{f}\in \mathrm{Hom}(F(X),G)$ s.t. $\hat{f}|_X = f$.

$$\begin{tikzcd} & F(X) \arrow[d, dashed, "\hat{f}"] \\ X \arrow[ur, hook, "i"] \arrow[r, "f"'] & G \end{tikzcd}$$

Pf: For $x\in X$, let $f_x: \langle x\rangle \to G$,

$$f_x(x^n) := f(x)^n.$$

Then $f_x\in \mathrm{Hom}(\langle x\rangle, G)$. So by the universal property of free product of groups, $\exists!\ \hat{f}\in \mathrm{Hom}(F(X),G)$ s.t. $\hat{f}|_{\langle x\rangle} = f_x$ for any x; and so $\hat{f}|_X = f$.

Uniqueness follows from the fact that X generates $F(X)$. ■