1 Homework 6.

1. For a group G, let $[G,G]$ be the group generated by $[g_1,g_2] := g_1g_2g_1^{-1}g_2^{-1}$'s where $g_1, g_2 \in G$. This is called the *derived subgroup* of G.

   (a) Prove that $[G,G]$ is a characteristic subgroup.

   (b) Prove that for a normal subgroup N of G, G/N is abelian precisely when $[G,G] \subseteq N$.

   (c) Prove that $[S_n, S_n] = A_n$ for every integer $n \geq 3$.

2. Suppose $n \geq 5$ and $m \geq 2$ are integers.

   (a) Find the composition factors of S_n.

   (b) Prove that if N is a non-trivial proper normal subgroup of S_n, then $N = A_n$.

   (c) Find out for what values of m, S_m is solvable.

3. Suppose the following is a SES

$$1 \to G_1 \to G_2 \to G_3 \to 1.$$

   Prove that G_2 is solvable if and only if G_1 and G_3 are solvable.

4. Prove that there is no finite group G such that $[G,G] \simeq S_4$.

   (**Hint.** Suppose to the contrary that there exists a finite group G such that $[G,G] \simeq S_4$. Convince yourself that

$$P := \{I, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

   is the unique Sylow 2-subgroup of A_4. Deduce that P is a characteristic subgroup of A_4. Consider the action of G on $[G,G] \simeq S_4$ by via conjugation. Since A_4 and P are characteristic subgroups of S_4, obtain an action by automorphisms on A_4/P. This gives you a group homomorphism from G to $\mathrm{Aut}(A_4/P)$. Argue why this implies that $[G,G]$ acts trivially on A_4/P. This means S_4 acts trivially on A_4/P by conjugations. Observe that

$$(1\ 2)(1\ 2\ 3)(1\ 2)P \neq (1\ 2\ 3)P,$$

   and get a contradiction.)

5. Prove that $D_\infty := \{ax + b \mid a \in \{\pm 1\}, b \in \mathbb{Z}\}$ under composition is an infinite solvable group which is generated by two elements of order 2. Find the center $Z(D_\infty)$ of D_∞.

(**Hint.** Think about the symmetries of the integer grid in the real line.)

6. For every group G, the group of outer automorphisms is

$$\mathrm{Out}(G) := \frac{\mathrm{Aut}(G)}{\mathrm{Inn}(G)}.$$

Let $\mathrm{Cl}(G)$ be the set of conjugacy classes of G.

(a) Prove that

$$(\theta\, \mathrm{Inn}(G)) \cdot [a] := [\theta(a)]$$

is a well-defined action of $\mathrm{Out}(G)$ on $\mathrm{Cl}(G)$, where $[g]$ is the conjugacy class of g in G.

(b) Argue why $f : \mathrm{Cl}(G) \to \mathbb{Z} \times \mathbb{Z}, f([g]) := (o(g), |[g]|)$ is fixed along an $\mathrm{Out}(G)$-orbit.

(c) Prove that $\mathrm{Aut}(S_n) \simeq \mathrm{Inn}(S_n)$ if $n \neq 6$.

(d) Prove that $\mathrm{Aut}(S_n) \simeq S_n$ if $n \neq 2, 6$.

(**Hint.** Use an argument similar to part (a) of problem 3 from HW 6.)