1 Homework 8.

1. Prove that $\langle a, b \mid [a, b] \rangle \simeq \mathbb{Z} \times \mathbb{Z}$.

   (**Hint.** First prove that there exists a surjective group homomorphism from $f : \langle a, b \mid [a, b] \rangle \to \mathbb{Z} \times \mathbb{Z}$. Then consider

   $$g : \mathbb{Z} \times \mathbb{Z} \to \langle a, b \mid [a, b] \rangle, \quad g(m, n) := a^n b^m.)$$

2. Suppose X_1 and X_2 are two disjoint sets. Prove that

   $$\langle X_1 \mid R_1 \rangle * \langle X_2 \mid R_2 \rangle \simeq \langle X_1 \cup X_2 \mid R_1 \cup R_2 \rangle.$$

   (**Hint.** Define $\theta_i : \langle X_i \mid R_i \rangle \to \langle X_1 \cup X_2 \mid R_1 \cup R_2 \rangle$ such that $\theta_i|_{X_i}$ is the identity function. Then use the universal property of free product of groups and obtain a surjective group homomorphism

   $$\theta : \langle X_1 \mid R_1 \rangle * \langle X_2 \mid R_2 \rangle \to \langle X_1 \cup X_2 \mid R_1 \cup R_2 \rangle.)$$

   Use the universal property of free groups and obtain a surjective group homomorphism

   $$\phi : \langle X_1 \cup X_2 \mid R_1 \cup R_2 \rangle \to \langle X_1 \mid R_1 \rangle * \langle X_2 \mid R_2 \rangle.$$

   Observe that $\phi \circ \theta = \mathrm{id}$ and $\theta \circ \phi = \mathrm{id}$.)

3. Prove that the subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ which is generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is isomorphic to $\langle a, b \mid b^2 \rangle$.

   (**Hint.** We have proved that this group is isomorphic to $\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z})$. Use this result together with the previous problem.)

4. Prove that $\mathrm{PSL}_2(\mathbb{Z}) \simeq \langle a, b \mid a^2, b^3 \rangle$.

   (**Hint.** In this problem, you are allowed to use the fact that $\mathrm{PSL}_2(\mathbb{Z})$ is generated by the cosets of

   $$\sigma := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tau := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Consider the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{R} \cup \{\infty\}$ by Möbius transformation. Let $\omega := \sigma\tau$. Notice that $w^3 = 1$. Show that $G_1 := \langle \tau \rangle$ and $G_2 := \langle \omega \rangle$ *play ping-pong* with the *table*, $X_1 := (-\infty, 0]$ and $X_2 := (0, \infty) \cup \{\infty\}$.)

5. Prove that the group of Euclidean symmetries of the integer grid is isomorphic to $\langle a, b \mid a^2, b^2 \rangle$.

   .(**Hint.** Use can use without proof that the group G of Euclidean symmetries is generated by $f : \mathbb{Z} \to \mathbb{Z}, f(x) := -x$ and $g : \mathbb{Z} \to \mathbb{Z}, g(x) := x + 1$. Consider f and $g \circ f$ and obtain a surjective group homomorphism from $\langle a, b \mid a^2, b^2 \rangle$ to G. Show that

$$\langle a, b \mid a^2, b^2 \rangle = \{(ab)^i \mid i \in \mathbb{Z}\} \cup a\{(ab)^i \mid i \in \mathbb{Z}\}.)$$

6. Let $G_n := \langle s_1, \ldots, s_{n-1} \mid s_i^2, (s_i s_j)^2 \text{ if } |i - j| > 1, (s_i s_{i+1})^3 \rangle$. Prove that $G_n \simeq S_n$.

   (**Hint**. Let $\tau_i := (i\, i+1)$ for $i \in [1..n-1]$. Prove that there exists a surjective group homomorphism $\phi : G_n \to S_n$ such that $\phi(s_i) = \tau_i$ for all $i \in [1..n-1]$. Use the following steps and show by induction n that $|G_n| \leq n!$.

   (a) Let H_{n-1} be the subgroup of G_n which is generated by $s_1, \ldots, s_{n-2}$. Show that there exists a surjective group homomorphism from G_{n-1} to H_{n-1}. By the induction hypothesis, obtain that $|H_{n-1}| \leq (n-1)!$.

   (b) Let $X_n := \{H_{n-1}, s_1 H_{n-1}, \ldots, s_1 \cdots s_{n-1} H_{n-1}\} \subseteq G_n / H_{n-1}$. Show that for every $i \in [1..n-1]$, $s_i X_n = X_n$. Deduce that $X_n = G_n / H_{n-1}$. Obtain that $[G_n : H_{n-1}] \leq n$, and so $|G_n| \leq n!$.)