Homework 1

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1. Let n be a square-free integer more than 3. Let

$$\mathbb{Z}[\sqrt{-n}] = \{a+\sqrt{-n}\, b \mid a, b \in \mathbb{Z}\}.$$

(a) Prove that 2, $\sqrt{-n}$, $1 \pm \sqrt{-n}$ are all irreducible in $\mathbb{Z}[\sqrt{-n}]$.

(b) Find an element in R which is irreducible and not prime

(c) Show that $\mathbb{Z}[\sqrt{-n}]$ is not a UFD.

2. Suppose p is an odd prime number. Show that the following are equivalent.

(a) p is not irreducible in $\mathbb{Z}[i]$.

(b) $\exists\, a, b \in \mathbb{Z}$, $p = a^2 + b^2$.

(c) $x^2 \equiv -1 \pmod{p}$ has a solution.

3. Let D be a UFD and F be its field of fractions.

(a) Suppose $\frac{r}{s}$ is a zero of $a_n x^n + \cdots + a_1 x + a_0$ where $r, s \in D$ and $\gcd(r,s) = [1]$. Prove that $r \mid a_0$ and $s \mid a_n$.

In particular, if $a \in F$ is a zero of a monic poly. in $D[x]$, then $a \in D$. (We say a UFD is <u>integrally closed</u>.)

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(b) Show that $\mathbb{Z}[2\sqrt{2}] = \{a + 2\sqrt{2}\, b \mid a, b \in \mathbb{Z}\}$ is not a UFD. (<u>Hint.</u> $(\sqrt{2})^2 - 2 = 0$.)

4. Let $A = \mathbb{Z} + x\mathbb{Q}[x] = \{a_n x^n + \cdots + a_1 x + a_0 \mid a_0 \in \mathbb{Z}, a_1, \ldots, a_n \in \mathbb{Q}\}$.

(a) Show that $f(x) \in A$ is irreducible $\Leftrightarrow$

either $f(x) = \pm p$ where p is a prime number

or $f(x)$ is irred. in $\mathbb{Q}[x]$ and $f(0) = \pm 1$.

(b) Show that $x \in A$ cannot be written as a product of finitely many irreducibles in A. Thus A is not a UFD.

(c) We proved in class that, if an integral domain is Noetherian, then any non-zero element can be written as a product of irreducible. And A has an ideal that is not finitely generated. Find an explicit ideal $\mathfrak{a} \triangleleft A$ that is not finitely generated.

5. A <u>Bezout</u> domain is an integral domain D in which

$\forall a, b \in D, \ \exists c \in D$ s.t. $\langle a, b \rangle = \langle c \rangle$.

(a) Prove that an integral domain D is a Bezout domain

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I have used HW assignments by Professor Rogalski and Professor Rhoades

if and only if $\forall\ a, b \in D \setminus \{0\}$ $\exists\, d \in D$ s.t.

(i) d is a g.c.d. of a and b.

(ii) $d \in \langle a, b \rangle$.

(b) Prove that every finitely generated ideal of a Bezout domain is principal (In particular a Noetherian Bezout domain is a PID.)

(c) Prove that D is a PID if and only if it is both a UFD and a Bezout domain

(Hint. For $0 \neq \mathfrak{a} \lhd D$, let $a \in \mathfrak{a}$ be an element with smallest number of irreducible factors.

$\forall\, b \in D$, show $\langle a, b \rangle = \langle a \rangle$.)

6. Let $f(x) \in (\mathbb{Z}/p\mathbb{Z})[x]$ be a polynomial of degree n. Prove that $(\mathbb{Z}/p\mathbb{Z})[x] / \langle f(x) \rangle$ has p^n elements. (Hint. Use division algorithm.)

7. Let A be the subring of $\mathbb{Q}[x, y]$ which is generated by $x, xy, xy^2, \ldots$; that means $A = \mathbb{Q}[x, xy, xy^2, xy^3, \ldots]$.
Prove that A is NOT Noetherian.