1. Let $n$ be a square-free integer more than 3. Let

$$\mathbb{Z}[\sqrt{-n}] = \{a + \sqrt{-n} b \mid a, b \in \mathbb{Z}\}.$$ 

(a) Prove that $2, \sqrt{-n}, 1 \pm \sqrt{-n}$ are all irreducible in $\mathbb{Z}[\sqrt{-n}]$.

(b) Find an element in $R$ which is irreducible and not prime.

(c) Show that $\mathbb{Z}[\sqrt{-n}]$ is not a UFD.

2. Suppose $p$ is an odd prime number. Show that the following are equivalent.

(a) $p$ is not irreducible in $\mathbb{Z}[i]$.

(b) $\exists a, b \in \mathbb{Z}, \ p = a^2 + b^2$.

(c) $x^2 \equiv -1 \pmod{p}$ has a solution.

3. Let $D$ be a UFD and $F$ be its field of fractions.

(a) Suppose $\frac{r}{s}$ is a zero of $a_n x^n + \ldots + a_1 x + a_0$ where $r, s \in D$ and $\gcd(r, s) = 1$. Prove that $r \mid a_0$ and $s \mid a_n$.

In particular, if $a \in F$ is a zero of a monic poly. in $D[x]$, then $a \in D$. (we say a UFD is integrally closed.)
(b) Show that \( \mathbb{Z} \left[ 2 \sqrt{2} \right] = \{ a + 2 \sqrt{2} b \mid a, b \in \mathbb{Z} \} \) is not a UFD. (Hint: \( (2) \left[ \frac{2}{2} \right] = 0 \).)

4. Let \( A = \mathbb{Z} + x \mathbb{Q}[x] = \{ \sum a_n x^n + a_0 x + a_0 \mid a_0, a_1, \ldots, a_n \in \mathbb{Q} \} \).

(a) Show that \( f(x) \in A \) is irreducible \( \iff \)

- either \( f(x) = \pm \) where \( \pm \) is a prime number
- or \( f(x) \) is irreducible in \( \mathbb{Q}[x] \) and \( f(a) = \pm 1 \).

(b) Show that \( x \in A \) cannot be written as a product of finitely many irreducibles in \( A \). Thus \( A \) is not a UFD.

(c) We proved in class that, if an integral domain is Noetherian, then any non-zero element can be written as a product of irreducible. And \( A \) has an ideal that is not finitely generated.

Find an explicit ideal \( A \) that is not finitely generated.

5. A **Bezout** domain is an integral domain \( D \) in which

\[ \forall a, b \in D, \exists c \in D \text{ s.t. } \langle a, b \rangle = \langle c \rangle. \]

(a) Prove that an integral domain \( D \) is a **Bezout** domain
if and only if $\forall a, b \in D \exists d \in D$ s.t.

(i) $d$ is a g.c.d. of $a$ and $b$.
(ii) $d \in \langle a, b \rangle$.

(b) Prove that every finitely generated ideal of a Bezout domain is principal (in particular a Noetherian Bezout domain is a PID).

(c) Prove that $D$ is a PID if and only if it is both a UFD and a Bezout domain.

(Hint. For $0 \neq r \in D$, let $a \in D$ be an element with smallest number of irreducible factors.

$\forall b \in D$, show $\langle a, b \rangle = \langle a \rangle$.)

6. Let $f(x) \in \mathbb{Z}/p\mathbb{Z}[x]$ be a polynomial of degree $n$. Prove that $\mathbb{Z}/p\mathbb{Z}[x]/\langle f(x) \rangle$ has $p^n$ elements. (Hint. Use division algorithm.)

7. Let $A$ be the subring of $\mathbb{Q}[x, y]$ which is generated by $x, xy, xy^2, \ldots$; that means $A = \mathbb{Q}[x, xy, xy^2, xy^3, \ldots]$. Prove that $A$ is not Noetherian.