Homework 3

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$\boxed{1}$ (Opposite ring) (a) Let R be a unital ring. Prove that

$\mathrm{End}_R(R) \simeq R^{op}$ as rings.

(b) Suppose $\exists\, \tau : R \to R$ s.t.
- $\tau(x+y) = \tau(x) + \tau(y)$
- $\tau(xy) = \tau(y)\tau(x)$
- $\tau(\tau(x)) = x$.

Prove that $R \simeq R^{op}$.

(c) Prove that $M_n(R)^{op} \simeq M_n(R^{op})$; in particular

$M_n(R)^{op} \simeq M_n(R)$ if R is commutative.

(d) Suppose G is a group and $\mathbb{C}G$ is the group ring of G over $\mathbb{C}$. Prove that $\mathbb{C}G^{op} \simeq \mathbb{C}G$.

(e) [NOT part of HW assignment] Can you find a ring R such that $R \not\simeq R^{op}$?

$\boxed{2}$ (Torsion submod.) Suppose R is an integral domain and M is an R-mod. Let $\mathrm{Tor}(M) := \{ m \in M \mid \exists\, r \in R \setminus \{0\},\ rm = 0 \}$.

(a) Prove that $\mathrm{Tor}(M)$ is a submod. of M.

(b) $\mathrm{Tor}(M/\mathrm{Tor}(M)) = 0$. ($M/\mathrm{Tor}(M)$ is torsion-free.)

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$\boxed{3.}$ (Annihilator) Let R be a unital ring and M be an R-mod.

The <u>annihilator $\mathrm{Ann}(M)$</u> of M is

$$\mathrm{Ann}(M) := \{r \in R \mid \forall m \in M,\ r.m = 0\}.$$

And the <u>annihilator $\mathrm{Ann}(m)$</u> of an element m of M is

$$\mathrm{Ann}(m) := \{r \in R \mid r.m = 0\}.$$

(Notice that $\mathrm{Ann}(M) = \bigcap_{m \in M} \mathrm{Ann}(m)$.)

(a) Prove that $\mathrm{Ann}(m)$ is a left ideal.

(b) Give an example where $\mathrm{Ann}(m)$ is NOT a two-sided ideal. (<u>Hint</u>. For instance think about $M_n(\mathbb{C})$; you have seen before that ideals of $M_n(R)$ are of the form $M_n(I)$ where $I \trianglelefteq R$; in particular $M_n(D)$ does not have a non-trivial two sided ideal if D is a division ring.)

(c) Prove that $\mathrm{Ann}(M)$ is a (both sided) ideal of R.

(d) Find a $\mathbb{Z}$-mod. M s.t. $\mathrm{Ann}(M) = 0$ and $\forall m \in M$, $\mathrm{Ann}(m) \neq 0$.

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(*Remark*. We say M is a faithful R-mod if $\mathrm{Ann}(M)=0$.)

(e) An R-mod M is an $R/_{\mathrm{Ann}(M)}$- mod w.r.t.

$$(r+\mathrm{Ann}(M))\cdot x := r\cdot x \quad \text{scalar multiplication.}$$

4. Let R be a unital ring, $I \lhd R$, and M be an R-mod.

Let $IM := \{\sum_{i=1}^{m} r_i\cdot x_i \mid r_i \in I,\ x_i \in M\}$.

(a) Prove that IM is a submodule of M.

(b) Prove that $I \subseteq \mathrm{Ann}(M/_{IM})$; and deduce that $M/_{IM}$ can be viewed as an $R/_I$-mod via

$$(r+I)\cdot(x+IM) := rx + IM.$$

5. Let $V=\bigoplus_{i=1}^{\infty}\mathbb{C}v_i$ be a countable dimensional vector space over $\mathbb{C}$. Let $R := \mathrm{End}_{\mathbb{C}}(V)$. Prove that as R-modules $R \simeq R\oplus R$.

(*Hint*. Use "projection" to odd and even components (or any other partition to two infinite sets.))

6. Let G be a group, and M be an abelian group. Give an explicit bijection between the set of linear G-actions on M and $\mathbb{Z}G$-module structures on M.

(You can use without proof that a linear G-action on M is given by $\mathrm{Hom}(G, \mathrm{Aut}(M))$ (group homomorphisms).)

$\boxed{7.}$ Let k be a field. For a subset A of the ring $k[x_1,\dots,x_n]$ of polynomials. Let $V(A) := \{\vec{v} \in k^n \mid \forall f(x_1,\dots,x_n) \in A, f(\vec{v}) = 0\}$.

And, for a subset X of k^n, let

$$I(X) := \{f \in k[x_1,\dots,x_n] \mid \forall \vec{v} \in X,\ f(\vec{v}) = 0\}.$$

(a) Prove that $I(X) \triangleleft k[x_1,\dots,x_n]$.

(b) Prove that, $\forall \emptyset \neq A \subseteq k[x_1,\dots,x_n], V(I(V(A))) = V(A)$.

(c) Prove that for any $\emptyset \neq A \subseteq k[x_1,\dots,x_n]$ there are finitely many polynomials $f_1,\dots,f_m$ s.t.

$$V(A) = V(f_1, f_2, \dots, f_m).$$

(d) Suppose $I \triangleleft k[x_1,\dots,x_n]$. Prove that $\sqrt{I} \subseteq I(V(I))$, where $\sqrt{I} := \{f \in k[x_1,\dots,x_n] \mid \exists m, f^m \in I\}$.