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Reading before Problem

Suppose $A$ is a unital commutative ring, $S \subseteq A$ is multiplicatively closed, and $M$ is an $A$-module. We can localize $M$ with respect to $S$ as we did $A$. Namely on $\underbrace{M}_{\text{(nuemerator)}} \times \underbrace{S}_{\text{(denomin.)}}$ we define the following relation:

$$(m_1, s_1) \sim (m_2, s_2) \iff \exists\, s \in S \text{ s.t. } s.(s_1 \cdot m_2 - s_2 \cdot m_1) = 0.$$

Convince yourself that $\sim$ is an equivalency relation on $M \times S$, and let $\frac{m}{s} := [(m,s)]$, and $S^{-1}M := \{\frac{m}{s} \mid m \in M, s \in S\}$.

Let $\frac{m_1}{s_1} + \frac{m_2}{s_2} := \frac{s_2 m_1 + s_1 \cdot m_2}{s_1 s_2}$ ; convince yourself that it is a well-defined operation and $(S^{-1}M, +)$ is an abelian group.

For $\frac{a}{s} \in S^{-1}A$ and $\frac{m}{s'} \in S^{-1}M$, let $\frac{a}{s} \cdot \frac{m}{s'} := \frac{a \cdot m}{ss'}$.

Convince yourself that it is well-defined, and it makes $S^{-1}M$ an $S^{-1}A$-mod.

For $\mathfrak{p} \in \mathrm{Spec}(A)$, we let $M_{\mathfrak{p}} := S_{\mathfrak{p}}^{-1} M$ where $S_{\mathfrak{p}} := A \setminus \mathfrak{p}$.

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1. (a) Suppose $M$ is an $A$-mod. Prove that

$$M=0 \iff \forall \mathfrak{p}\in \mathrm{Spec}(A),\ M_{\mathfrak{p}}=0$$
$$\iff \forall \mathfrak{m}\in \mathrm{Max}(A),\ M_{\mathfrak{m}}=0.$$

(Hint. Clearly the only non-trivial part is: $\forall \mathfrak{m}\in \mathrm{Max}(A), M_{\mathfrak{m}}=0 \overset{?}{\Rightarrow} M=0$.
For $x\in M$, consider $\mathrm{ann}(x)$; and show it cannot be proper.)

(b) Let $\phi: M_1\to M_2$ be an $A$-mod. homomorphism. And $S$ is a multiplicatively closed subset of $A$. Let $S^{-1}\phi: S^{-1}M_1\to S^{-1}M_2$,

$$(S^{-1}\phi)\left(\frac{m_1}{s}\right) := \frac{\phi(m_1)}{s}.$$

Show that $S^{-1}\phi$ is a well-defined $S^{-1}A$-mod. homomorphism.

(For $\mathfrak{p}\in \mathrm{Spec}(A)$, we let $\phi_{\mathfrak{p}} := S_{\mathfrak{p}}^{-1}\phi$ where $S_{\mathfrak{p}} := A\setminus \mathfrak{p}$)

Suppose $M_1$ is a submodule of $M_2$. Observe that $S^{-1}M_1$ is a submod of $S^{-1}M_2$. Convince yourself that $S^{-1}M_2/S^{-1}M_1 \simeq S^{-1}(M_2/M_1)$.

(c) Let $\phi: M_1\to M_2$ be an $A$-mod. homomorphism. Prove that

$$\phi \text{ is injective} \iff \forall \mathfrak{m}\in \mathrm{Max}(A),\ \phi_{\mathfrak{m}} \text{ is injective.}$$

(Hint. Show that $\ker(\phi_{\mathfrak{m}}) = (\ker(\phi))_{\mathfrak{m}}$.)

(d) Show that $\phi$ is surjective $\iff \forall \mathfrak{m}\in \mathrm{Max}(A)$, $\phi_{\mathfrak{m}}$ is surjective.

(Hint. Consider the co-kernel of $\phi$; that means $M_2/\mathrm{Im}\,\phi$.
And co-kernel of $\phi_{\mathfrak{m}}$'s.)

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$\boxed{\text{Reading before problem.}}$ Suppose $A$ is a unital commutative ring and $S$ is a multiplicatively closed set. As we have seen in problem 1, if $\mathfrak{a} \lhd A$, then $S^{-1}\mathfrak{a} \lhd S^{-1}A$; and $S^{-1}(A/\mathfrak{a}) \simeq S^{-1}A/S^{-1}\mathfrak{a}$ as $S^{-1}A$-modules. Convince yourself that this implies

$$\overline{S}^{-1}(A/\mathfrak{a}) \simeq S^{-1}A/S^{-1}\mathfrak{a}$$

as rings where $\overline{S} = \{s+\mathfrak{a} \in A/\mathfrak{a} \mid s \in S\}$.

2.(a) Suppose $\tilde{\mathfrak{a}}$ is an ideal of $S^{-1}A$. Let

$$\mathfrak{a} := \{a \in A \mid \tfrac{a}{1} \in \tilde{\mathfrak{a}}\}.$$

Prove that $\mathfrak{a} \lhd A$ and $\tilde{\mathfrak{a}} = S^{-1}\mathfrak{a}$.

(b) Let $\mathcal{O}_S := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{p} \cap S = \emptyset\}$. Let

$\Phi : \mathcal{O}_S \longrightarrow \operatorname{Spec}(S^{-1}A)$, $\Phi(\mathfrak{p}) := S^{-1}\mathfrak{p}$, and

$\Psi : \operatorname{Spec}(S^{-1}A) \longrightarrow \mathcal{O}_S$, $\Psi(\tilde{\mathfrak{p}}) := \{a \in A \mid \tfrac{a}{1} \in \tilde{\mathfrak{p}}\}$.

Prove that $\Phi$ and $\Psi$ are well-defined and they are inverse of each other. (and so there is a bijection between prime ideals of $S^{-1}A$ and prime ideals of $A$ that

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do not intersect $S$.)

(Explanation. • You have to show $S^{-1}\mathfrak{p}$ is prime if $\mathfrak{p}$ is.

Think about $S^{-1}A/_{S^{-1}\mathfrak{p}} \simeq \bar{S}^{-1}(A/_{\mathfrak{p}}) \hookrightarrow$ field of fractions of $A/\mathfrak{p}$

• Next you have to show

$$\left.\begin{array}{l}\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec}(A) \\ S^{-1}\mathfrak{p}_1 = S^{-1}\mathfrak{p}_2\end{array}\right\} \overset{?}{\Longrightarrow} \mathfrak{p}_1 = \mathfrak{p}_2.\ )$$

3. (a) Suppose $A$ is a unital commutative ring and $\mathfrak{m} \triangleleft A$.

Prove that $\operatorname{Max}(A) = \{\mathfrak{m}\}$ if and only if $A^{\times} = A \setminus \mathfrak{m}$.

(Such a ring is called a <u>local ring</u>.)

(b) Suppose $A$ is a unital commutative ring. Prove that $A_{\mathfrak{p}}$ is a local ring for any $\mathfrak{p} \in \operatorname{Spec}(A)$.

$\boxed{\text{Reading before problem}}$ Determinant can be defined for matrices with entries in a <u>unital commutative ring</u>:

$$\det [a_{ij}] := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}, \quad \text{where}$$

$S_n$ is the symmetric group, and $\operatorname{sgn}: S_n \to \{\pm 1\}$ is the sign

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group homomorphism. Similar to the $n\times n$ matrices over a field, one can define <u>minors</u> of $x=[a_{ij}]$.

The $l,k$-minor of $x=[a_{ij}]$ is the determinant of the $(n-1)\times(n-1)$ matrix $x(l,k)$ that one gets after removing the $l^{th}$ row and the $k^{th}$ column.

$$x(l,k) := \begin{bmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{l1} & \cdots & a_{lk} & \cdots & a_{ln} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nk} & \cdots & a_{nn} \end{bmatrix}$$

Similar to Cramer's rule, we can define the adjunct matrix $\mathrm{adj}(x)$ of $x$. The $(i,j)$-entry of $\mathrm{adj}(x)$ is $(-1)^{i+j}\det x(j,i)$.

Here are the main properties of $\det: M_n(A)\to A$.

(1) det is multi-linear with respect to columns.

(1′) det is multi-linear with respect to rows.

(2) $\det(I)=1$.

(3) If $x$ has two identical rows, then $\det x=0$

(3′) If $x$ has two identical columns, then $\det x=0$

(4) $\mathrm{adj}(x)\cdot x = x\cdot \mathrm{adj}(x) = \det(x)\, I$.

(5) $\forall\, x,y\in M_n(A)$, $\det(xy)=\det(x)\det(y)$.

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4. (a) Suppose $A$ is a unital commutative ring, and $GL_n(A) := M_n(A)^{\times}$.

Prove that $x \in GL_n(A) \iff \det x \in A^{\times}$.

(b) Suppose $A$ is a unital commutative ring and $\mathrm{Max}(A) = \{\mathfrak{m}\}$.

Suppose $\phi : A^n \to A^n$ is an $A$-mod. homomorphism and let

$x_\phi \in M_n(A)$ be its associated matrix. Convince yourself that

$\phi$ is a bijection if and only if $x_\phi \in GL_n(A)$.

Prove the following statements are equivalent:

(1) $\phi : A^n \to A^n$ is surjective.

(2) $\overline{\phi} : (A/\mathfrak{m})^n \to (A/\mathfrak{m})^n$ is bijective, where $\overline{\phi}$ is induced by $\phi$.

(3) $\phi : A^n \to A^n$ is bijective.

(Hint. Show (1) $\iff$ (2) and (2) $\iff$ (3). Use linear algebra to show $\det(\phi) \notin \mathfrak{m}$.)

(c) Suppose $A$ is a unital commutative ring, and $\phi : A^n \to A^n$ is an $A$-mod. homomorphism. Prove that

$$\phi \text{ is surjective} \iff \phi \text{ is bijective.}$$

(Hint. Use Problem 1.c, 1.d, 3.b)

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5. Suppose $A$ is a unital commutative ring, and $\phi: A^n \to A^m$ is surjective. Prove that $n \geq m$.

$\boxed{\text{Reading before problem}}$ A module $M$ is called <u>Noetherian</u> if the following (equivalent) statements hold:

(a) Any chain $(N_i)_{i\in I}$ of submodules of $M$ has a maximum.

(b) Any non-empty set $\Sigma$ of submodules of $M$ has a maximal element.

(c) $M$ satisfies the ascending chain condition (a.c.c.); that means if $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ are submodules of $M$, then $\exists\, i_0$ s.t. $N_{i_0} = N_{i_0+1} = \cdots$.

(d) All the submodules of $M$ are finitely generated.

Go over <u>Lecture 28 of math 200a</u> and see that similar arguments imply (a), (b), (c), and (d) are equivalent.

Observe that $A$ is a Noetherian ring if and only if $A$ is a Noetherian $A$-mod.

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6. (a) Suppose $N$ is a submodule of $M$. Prove that

$$M \text{ is Noetherian} \iff N \text{ and } M/N \text{ are Noetherian.}$$

(b) Suppose $A$ is a Noetherian ring, and $M$ is a finitely generated $A$-module. Prove that $M$ is Noetherian.

7. (a) Suppose $A$ is a Noetherian unital commutative ring, and $\phi: A^n \longrightarrow A^m$ is injective. Prove that $n \leq m$.

(Hint. If not, $\phi(A^n) \oplus A^{n-m} \subseteq A^n$

explain

$\Rightarrow \phi^2(A^n) \oplus \phi(A^{n-m}) \oplus A^{n-m} \subseteq A^n \Rightarrow \cdots$

$\phi^i(A^n) \oplus \phi^{i-1}(A^{n-m}) \oplus \phi^{i-2}(A^{n-m}) \oplus \cdots \oplus A^{n-m} \subseteq A^n$.

$\Rightarrow A^{n-m} \subsetneq A^{n-m} \oplus \phi(A^{n-m}) \subsetneq \cdots \subseteq A^n$.)

(b) Suppose $A$ is a unital commutative ring, and $\phi: A^n \to A^m$ is injective. Prove that $n \leq m$.

(Hint. Suppose $x_\phi = [a_{ij}]$ is the associated matrix; and let $A_0$ be the subring of $A$ which is generated by $a_{ij}$'s. Consider $\phi|_{A_0^n}$, discuss why $\phi|_{A_0^n}: A_0^n \to A_0^m$ is injective. Use Hilbert's basis theorem and part (a).)

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(c) Suppose $A$ is a unital commutative ring, and $M$ is a finitely generated $A$-module. Let

$d(M) :=$ minimum number of generators of $M$, and

$\operatorname{rank}(M) :=$ maximum number of linearly independent elements of $M$.

Prove that $\operatorname{rank}(M) \le d(M)$.

(Hint. Let $d(M) = n$ and $\operatorname{rank}(M) = m$. Then

$\exists\, \phi : A^n \to M$ surjective and $\exists\, \psi : A^m \to M$ injective.

So, for any $1 \le i \le m$, $\exists\, v_i \in A^n$ s.t. $\phi(v_i) = \psi(e_i)$.

Let $\theta(e_i) := v_i$ and extend it to an $A$-mod. homomorphism

$\theta : A^m \to A^n$ s.t.

$$\begin{tikzcd} A^m \arrow[d, dashed, "\theta"'] \arrow[dr, "\psi"] & \\ A^n \arrow[r, "\phi"'] & M \end{tikzcd}$$

Deduce that $\theta$ is injective. )

(Remark. In class, we discussed the case where $A$ is an integral domain.)