1. Prove that $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{3}]$ are not isomorphic.

2. Prove that $\mathbb{F}_p(x,y)/\mathbb{F}_p(x^p,y^p)$ is not a simple extension.

3. Suppose $f(x) \in \mathbb{Q}[x]$ is irreducible, $\deg f = p$ is prime, and $f$ has $p-2$ real and 2 non-real zeros in $\mathbb{C}$. Let $K$ be a splitting field of $f(x)$ over $\mathbb{Q}$. Prove that $\text{Aut}(K/\mathbb{Q}) \cong S_p$.

(Hint: Since $K/\mathbb{Q}$ is normal, complex conjugation gives us an element of $\text{Aut}(K/\mathbb{Q})$. Let $\alpha \in K$ be a zero of $f$. Then $\phi = [\mathbb{Q}[x]:\mathbb{Q}] | [K:\mathbb{Q}] = |\text{Aut}(K/\mathbb{Q})|$. So $\text{Aut}(K/\mathbb{Q})$ has an element of order $p$. Now think about the action of $\text{Aut}(K/\mathbb{Q})$ on the set of zeros of $f(x)$.)

4. Let $E/F$ be an algebraic extension. Let

$$E_{\text{sep}} = \{ \alpha \in E \mid m_{\alpha,F}(x) \text{ is separable} \}.$$

Prove that (1) $E_{\text{sep}}$ is a field and $E_{\text{sep}}/F$ is a separable extension.

(2) If $\text{char}(F) = p > 0$, then $\forall \alpha \in E$, $\exists k \in \mathbb{Z}^+$, $m_{\alpha,E_{\text{sep}}}(x) = x^k - \alpha^k$.

In particular $\alpha^k \in E_{\text{sep}}$. 
(Hint. Suppose \( \alpha \in \mathbb{C} \). Let \( L \) be a splitting field of \( m_{\alpha,F}(x) = m_{\alpha,K}(x) \) over \( F \). Argue that \( L/F \) is separable. Deduce \( \alpha \in \mathbb{C} \) is separable.

- \( m_{\alpha,F}(x) = m_{\alpha,K}(x) \) where \( m_{\alpha,K}(x) \) is irreducible and separable.

Deduce that \( \alpha \in K_{\text{sep}} \).

5. Suppose \( E/F \) is a normal extension. Let \( E_{\text{sep}} \) be as in Problem 4.

Prove that \( E_{\text{sep}}/F \) is a Galois extension.

6. Suppose \( F \subseteq E \subseteq K \) is a tower of algebraic field extensions. Prove

\[
K/F \text{ is separable } \iff K/E \text{ and } E/F \text{ are separable.}
\]

(Hint. Consider \( K_{\text{sep}} \) as in problem 4. Deduce \( K_{\text{sep}} \supseteq E \); and so

\( \forall \alpha \in K, m_{\alpha,K_{\text{sep}}}(x) \mid m_{\alpha,E}(x) \). Now use part (2) of problem 4.)

7. Suppose \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Let \( F := \text{Fix}(\sigma) \). Suppose \( E/F \) is a finite Galois extension (for some subfield \( E \) of an algebraic closure \( \overline{\mathbb{Q}} \) of \( \mathbb{Q} \)). Prove that \( \text{Gal}(E/F) \) is a finite cyclic group.

8. Let \( E \subseteq \mathbb{C} \) be a splitting field of \( x^{p-2} \) over \( \mathbb{Q} \) where \( p \) is an odd prime. Prove that \( \text{Gal}(E/\mathbb{Q}) \cong \mathbb{Z}/p \mathbb{Z} \times (\mathbb{Z}/p \mathbb{Z})^x \).