

Lecture 09: direct sum

Thursday, January 25, 2018 11:53 PM

Def. Suppose $\{N_i\}_{i \in I}$ is a family of submod. of M .

We say $\sum_{i \in I} N_i$ is the internal direct sum of N_i 's if

$$\text{for } \sum_{i \in I} n_i, \sum_{i \in I} n_i' \in \sum_{i \in I} N_i,$$

$$\sum_{i \in I} n_i = \sum_{i \in I} n_i' \text{ implies } n_i = n_i' \text{ for any } i \in I.$$

In this case we write $\bigoplus_{i \in I} N_i$.

Def. Suppose $\{M_i\}_{i \in I}$ is a family of left R -mod.

Let $\bigoplus_{i \in I} M_i := \{ (m_i)_{i \in I} \mid m_i \in M_i, m_i \text{'s are 0 except for finitely many } i \}$

and $\prod_{i \in I} M_i := \{ (m_i)_{i \in I} \mid m_i \in M_i \}$.

$\bigoplus_{i \in I} M_i$ is called the external direct sum of M_i 's and

$\prod_{i \in I} M_i$ is called the direct product of M_i 's.

Ex. There is a bijection between $\bigoplus_{i \in \mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$ and set of

all the finite subsets of \mathbb{Z} , and there is a bijection

between $\prod_{i \in \mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$ and the power set $\mathcal{P}(\mathbb{Z})$ of \mathbb{Z} .

Use this to observe that $\bigoplus_{i \in \mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$ is countable and

$\prod_{i \in \mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$ is uncountable.

Lecture 09: Universal property of external direct sum

Monday, January 29, 2018 10:29 AM

Remark. It is more formal to write $\prod_{i \in I} N_i$ as

$$\{f: I \rightarrow \bigsqcup_{i \in I} N_i \mid \forall i \in I, f(i) \in N_i\}, \text{ and } \bigoplus_{i \in I} N_i \subseteq \prod_{i \in I} N_i.$$

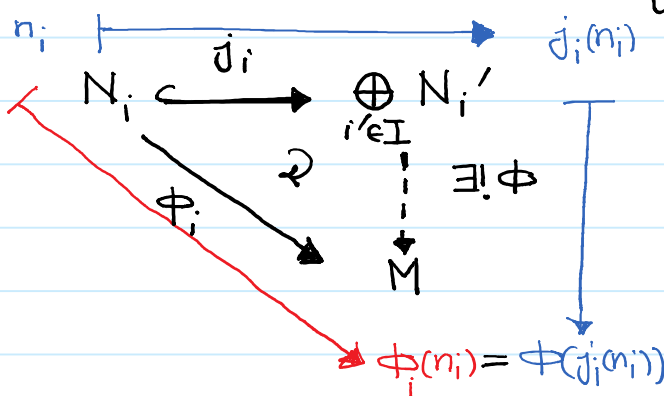
Universal Property of External Direct Sum.

Suppose $\{N_i\}_{i \in I}$ is a family of R -mod and M is an R -mod.

Suppose $\phi_i: N_i \rightarrow M$ is an R -mod. homomorphism. Then $\exists!$

R -mod homomorphism $\phi: \bigoplus_{i \in I} N_i \rightarrow M$ st. $\phi(j_i(n_i)) = \phi_i(n_i)$

where $j_i: N_i \rightarrow \bigoplus_{i' \in I} N_{i'}$, $(j_i(n_i))_{i'} := \begin{cases} n_i & \text{if } i = i' \\ 0 & \text{otherwise.} \end{cases}$



Pf. Existence. $\phi((n_i)) := \sum_{i \in I} \phi_i(n_i)$. Notice that, since

only finitely many terms are non-zero, the above sum is a finite summation.

$$\begin{aligned} \underline{R\text{-mod.}} \quad \phi((n_i) + r(n'_i)) &= \phi((n_i + r n'_i)) = \sum \phi_i(n_i + r n'_i) \\ &= \sum \phi_i(n_i) + r \phi_i(n'_i) = \sum \phi_i(n_i) + r \sum \phi_i(n'_i) \\ &= \phi((n_i)) + r \phi((n'_i)). \end{aligned}$$

Lecture 09: External sum and internal sum

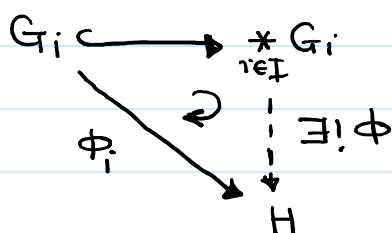
Friday, January 26, 2018 2:02 PM

Uniqueness Suppose $\phi \in \text{Hom}_{\mathbb{R}}(\bigoplus_{i \in I} N_i, M)$ and $\phi(j_i(n_i)) = \phi_i(n_i)$.

$$\begin{aligned} \text{Then } \phi((n_i)) &= \phi\left(\sum_i j_i(n_i)\right) = \sum \phi(j_i(n_i)) \\ &= \sum \phi_i(n_i). \end{aligned}$$

(Knowing ϕ on N_i 's, we linearly extend it.) ■

Remark. In group theory, the free product of G_i 's satisfy the above mentioned universal property:



Proposition. Suppose $\{N_i\}_{i \in I}$ is a family of submod. of M .

Then the following are equivalent:

(a) $\sum_{i \in I} N_i$ is an internal direct sum.

(b) $\forall j \in I, N_j \cap \sum_{i \in I, i \neq j} N_i = 0$

(c) $\bigoplus_{i \in I} N_i \xrightarrow{\phi} \sum_{i \in I} N_i, \phi((n_i)_{i \in I}) := \sum_{i \in I} n_i$ is an isomorphism.
(external direct sum)

Lecture 09: internal and external direct sums

Friday, January 26, 2018 12:11 AM

Pf. (a) \Rightarrow (b).

If not, $\exists n_j$ and $(n_i)_{i \in I \setminus \{j\}}$ s.t. $n_j = \sum_{i \in I \setminus \{j\}} n_i \neq 0$

which contradicts the assumption that $\sum_{i \in I} N_i$ is an internal direct sum.

(b) \Rightarrow (c). Since $(n_i)_{i \in I} \in \bigoplus N_i$ has only finitely many non-zero components, ϕ is well-defined. It is rather easy to check that ϕ is an R -mod. homomorphism. By the def. of $\sum_{i \in I} N_i$, we know that ϕ is surjective. (till this point we did not need any assumption. Now we are going to use (b) to show ϕ is injective. If not,

$$\exists (n_i)_{i \in I} \in \bigoplus_{i \in I} N_i \setminus \{0\} \text{ s.t. } \phi((n_i)_{i \in I}) = 0$$

$$\Leftrightarrow \sum_{i \in I} n_i = 0, \text{ and } \exists j \in I \text{ s.t. } n_j \neq 0.$$

$$\Rightarrow n_j = \sum_{i \in I \setminus \{j\}} (-n_i) \in N_j \cap \sum_{i \in I \setminus \{j\}} N_i \text{ which contradicts (b).}$$

$$(c) \Rightarrow (a). \text{ If } \sum_{i \in I} n_i = \sum_{i \in I} n'_i, \text{ then } \phi((n_i)_{i \in I}) = \phi((n'_i)_{i \in I})$$

$$\Rightarrow \text{injectivity implies } (n_i)_{i \in I} = (n'_i)_{i \in I};$$

$$\Rightarrow \forall i \in I, n_i = n'_i. \quad \blacksquare$$

Lecture 09: direct sum; free module

Friday, January 26, 2018 8:37 AM

Cor. If $\sum_{i \in I} N_i$ is an internal direct sum, then it is isomorphic to the external direct sum $\bigoplus_{i \in I} N_i$.

Def. We say a mod. M is said to be free on the subset A

if $M = \bigoplus_{a \in A} R a$ as an internal direct sum.

Proposition For any non-empty set A , there is a free left R -mod. $F(A)$ on the set A with the following

universal property:

$$\begin{array}{ccc} A & \xrightarrow{\quad} & F(A) \\ & \searrow \phi & \downarrow \hat{\phi} \\ & & M \end{array}$$

Pf of proposition. Let $F(A) := \bigoplus_{a \in A} R$ (external direct sum)

$= \{ (r_a)_{a \in A} \mid r_a \in R, \text{ zero except for finitely many } a \in A \}$.

Let M be an R -mod. and $\phi: A \rightarrow M$ be a function.

Let $\hat{\phi}: \bigoplus_{a \in A} R \rightarrow M$, $\hat{\phi}((r_a)_{a \in A}) := \sum_{a \in A} r_a \phi(a)$.

Since r_a 's are zero except for finitely many a 's, the above sum has only finitely many non-zero terms. It is rather easy to check that $\hat{\phi}$ is an R -mod. homomorphism.

Lecture 09: Free modules

Friday, January 26, 2018 8:50 AM

Let $j: A \rightarrow F(A)$, $j(a) = (r_{a'})_{a' \in A}$ where $r_{a'} = \begin{cases} 1 & \text{if } a=a' \\ 0 & \text{other} \\ \dots & \dots \end{cases}$

Then j is an embedding and $\hat{\phi}(j(a)) = \phi(a)$.

- $F(A)$ is generated by $j(A)$ as an R -mod.
- Suppose $\hat{\phi}: F(A) \rightarrow M$ is an R -mod; since $F(A)$ is generated by $j(A)$, $\hat{\phi}$ is uniquely determined by $\hat{\phi}|_{j(A)}$.

Proposition. Suppose R is a unital commutative ring. Suppose $m, n \in \mathbb{Z}^+$. If $R^n = R^m$, then $n = m$.

Remark. In your homework assignment, there is a problem which implies the above property does not necessarily hold for non-commutative rings: $\text{End}_{\mathbb{C}}(\bigoplus_{i \in \mathbb{Z}} \mathbb{C})$. It does hold for division rings.

Pf. Since R is a unital ring, it has a maximal ideal \mathfrak{m} .

Since R is a unital commutative ring, R/\mathfrak{m} is a field.

For any $k \in \mathbb{Z}^+$, $\mathfrak{m}R^k := \left\{ \sum_k m_i v_i \mid m_i \in \mathfrak{m}, v_i \in R^k \right\}$
 $= \mathfrak{m} \quad (?)$. (We will continue...)