Lecture 09: direct sum

Thursday, January 25, 2018 11:53 PM

<u>Def.</u> Suppose $\{N_i\}_{i\in I}$ is a family of submod. of M.

We say $\sum_{i\in I} N_i$ is the <u>internal direct sum</u> of N_i's if

for $\sum_{i\in I} n_i, \sum_{i\in I} n_i' \in \sum_{i\in I} N_i$,

$$\sum_{i\in I} n_i = \sum_{i\in I} n_i' \text{ implies } n_i = n_i' \text{ for any } i\in I.$$

In this case we write $\bigoplus_{i\in I} N_i$.

<u>Def.</u> Suppose $\{M_i\}_{i\in I}$ is a family of left R-mod.

Let $\bigoplus_{i\in I} M_i := \{(m_i)_{i\in I} \mid m_i\in M_i,\ m_i\text{'s are } 0 \text{ except for finitely many}\}$

and $\prod_{i\in I} M_i := \{(m_i)_{i\in I} \mid m_i \in M_i\}$.

$\bigoplus_{i\in I} M_i$ is called the external direct sum of M_i's and

$\prod_{i\in I} M_i$ is called the direct product of M_i's.

<u>Ex.</u> There is a bijection between $\bigoplus_{i\in\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$ and set of all the finite subsets of $\mathbb{Z}$, and there is a bijection between $\prod_{i\in\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$ and the power set $P(\mathbb{Z})$ of $\mathbb{Z}$.

Use this to observe that $\bigoplus_{i\in\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$ is countable and $\prod_{i\in\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$ is uncountable.

Lecture 09: Universal property of external direct sum

Monday, January 29, 2018 10:29 AM

<u>Remark</u>. It is more formal to write $\prod_{i\in I} N_i$ as

$\{f: I \to \coprod_{i\in I} N_i \mid \forall i\in I,\ f(i)\in N_i\}$, and $\bigoplus_{i\in I} N_i \subseteq \prod_{i\in I} N_i$.

<u>Universal Property of External Direct Sum.</u>

Suppose $\{N_i\}_{i\in I}$ is a family of R-mod and M is an R-mod.

Suppose $\phi_i : N_i \longrightarrow M$ is an R-mod. homomorphism. Then $\exists!$

R-mod homomorphism $\Phi: \bigoplus_{i\in I} N_i \to M$ s.t. $\Phi(j_i(n_i)) = \phi_i(n_i)$

where $j_i : N_i \to \bigoplus_{i'\in I} N_{i'}$, $(j_i(n_i))_{i'} := \begin{cases} n_i & \text{if } i=i' \\ 0 & \text{otherwise.} \end{cases}$

$$n_i \longmapsto j_i(n_i)$$

$$\begin{tikzcd}
N_i \arrow[r, hook, "j_i"] \arrow[dr, "\phi_i"'] & \bigoplus_{i'\in I} N_{i'} \arrow[d, dashed, "\exists!\Phi"] \\
& M
\end{tikzcd}$$

$$\phi_i(n_i) = \Phi(j_i(n_i))$$

<u>Pf</u>. <u>Existence</u>. $\Phi((n_i)) := \sum_{i\in I} \phi_i(n_i)$. Notice that, since only finitely many terms are non-zero, the above sum is a finite summation.

<u>R-mod</u>. $\Phi((n_i)+r(n_i')) = \Phi((n_i+rn_i')) = \sum \phi_i(n_i+rn_i')$

$= \sum \phi_i(n_i) + r\phi_i(n_i') = \sum \phi_i(n_i) + r\sum \phi_i(n_i')$

$= \Phi((n_i)) + r\Phi((n_i'))$.

Lecture 09: External sum and internal sum

Friday, January 26, 2018 2:02 PM

<u>Uniqueness</u> Suppose $\phi \in \mathrm{Hom}_R(\bigoplus_{i\in I} N_i, M)$ and $\phi(j_i(n_i)) = \phi_i(n_i)$.

Then $\phi((n_i)) = \phi(\sum_i j_i(n_i)) = \sum \phi(j_i(n_i))$

$$= \sum \phi_i(n_i).$$

(Knowing ϕ on N_i's, we linearly extend it.) ∎

<u>Remark</u>. In group theory, the free product of G_i's satisfy the above mentioned universal property:

$$\begin{tikzcd} G_i \arrow[r, hook] \arrow[dr, "\phi_i"'] & \ast_{i\in I} G_i \arrow[d, dashed, "\exists! \phi"] \arrow[dl, phantom, "\circlearrowleft"] \\ & H \end{tikzcd}$$

<u>Proposition</u>. Suppose $\{N_i\}_{i\in I}$ is a family of submod. of M.

Then the following are equivalent:

(a) $\sum_{i\in I} N_i$ is an internal direct sum.

(b) $\forall j \in I$, $N_j \cap \sum_{i\in I\setminus\{j\}} N_i = 0$

(c) $\bigoplus_{i\in I} N_i \xrightarrow{\phi} \sum_{i\in I} N_i$, $\phi((n_i)_{i\in I}) := \sum_{i\in I} n_i$ is an isomorphism.

(external direct sum)

Pf. (a) $\Rightarrow$ (b).

If not, $\exists\ n_j$ and $(n_i)_{i\in I\setminus\{j\}}$ s.t. $n_j = \sum_{i\in I\setminus\{j\}} n_i \neq 0$ which contradicts the assumption that $\sum_{i\in I} N_i$ is an internal direct sum.

(b) $\Rightarrow$ (c). Since $(n_i)_{i\in I} \in \bigoplus N_i$ has only finitely many non-zero components, ϕ is well-defined. It is rather easy to check that ϕ is an R-mod. homomorphism. By the def. of $\sum_{i\in I} N_i$, we know that ϕ is surjective. (till this point we did not need any assumption. Now we are going to use (b) to show ϕ is injective. If not,

$$\exists\ (n_i)_{i\in I} \in \bigoplus_{i\in I} N_i \setminus \{0\} \text{ s.t. } \phi((n_i)_{i\in I}) = 0$$

$$\Rightarrow \sum_{i\in I} n_i = 0, \text{ and } \exists\ j\in I \text{ s.t. } n_j \neq 0.$$

$$\Rightarrow n_j = \sum_{i\in I\setminus\{0\}} (-n_i) \in N_j \cap \sum_{i\in I\setminus\{j\}} N_i \text{ which contradicts (b).}$$

(c) $\Rightarrow$ (a). If $\sum_{i\in I} n_i = \sum_{i\in I} n_i'$, then $\phi((n_i)_{i\in I}) = \phi((n_i')_{i\in I})$

$$\Rightarrow \text{injectivity implies } (n_i)_{i\in I} = (n_i')_{i\in I};$$

$$\Rightarrow \forall i\in I, n_i = n_i'. \quad \blacksquare$$

Lecture 09: direct sum; free module

Friday, January 26, 2018 8:37 AM

Cor. If $\sum_{i\in I} N_i$ is an internal direct sum, then it is isomorphic to the external direct sum $\bigoplus_{i\in I} N_i$.

Def. We say a mod. M is said to be **free** on the subset A if $M = \bigoplus_{a\in A} Ra$ as an internal direct sum.

Proposition For any non-empty set A, there is a free left R-mod. $F(A)$ on the set A with the following universal property:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr, "\phi"'] & F(A) \arrow[d, dashed, "\hat{\Phi}"] \\ & M \end{tikzcd}$$

Pf of proposition. Let $F(A) := \bigoplus_{a\in A} R$ (external direct sum)

$= \{(r_a)_{a\in A} \mid r_a \in R, \text{ zero except for finitely many } a\in A\}$.

Let M be an R-mod. and $\phi: A \to M$ be a function.

Let $\hat{\Phi}: \bigoplus_{a\in A} R \to M$, $\hat{\Phi}((r_a)_{a\in A}) := \sum_{a\in A} r_a \phi(a)$.

Since r_a's are zero except for finitely many a's, the above sum has only finitely many non-zero terms. It is rather easy to check that $\hat{\Phi}$ is an R-mod. homomorphism.

Lecture 09: Free modules

Friday, January 26, 2018 8:50 AM

Let $j: A \to F(A)$, $j(a) = (r_{a'})_{a' \in A}$ where $r_{a'} = \begin{cases} 1 & \text{if } a = a' \\ 0 & \text{other} \end{cases}$

Then j is an embedding and $\hat{\Phi}(j(a)) = \phi(a)$.

- $F(A)$ is generated by $j(A)$ as an R-mod.
- Suppose $\hat{\Phi}: F(A) \to M$ is an R-mod; since $F(A)$ is generated by $j(A)$, $\hat{\Phi}$ is uniquely determined by $\hat{\Phi}|_{j(A)}$.

$\blacksquare$

<u>Proposition</u>. Suppose R is a unital commutative ring. Suppose $m, n \in \mathbb{Z}^+$. If $R^n \simeq R^m$, then $n = m$.

<u>Remark</u>. In your homework assignment, there is a problem which implies the above property does not necessarily hold for non-commutative rings: $\operatorname{End}_{\mathbb{C}}(\bigoplus_{i \in \mathbb{Z}} \mathbb{C})$. It does hold for division rings.

<u>Pf</u>. Since R is a unital ring, it has a maximal ideal $\mathfrak{m}$.

Since R is a unital commutative ring, $R/\mathfrak{m}$ is a field.

For any $k \in \mathbb{Z}^+$, $\mathfrak{m} R^k := \{ \sum_k m_i v_i \mid m_i \in \mathfrak{m}, v_i \in R^k \}$
$= \mathfrak{m}^k$ (?). (We will continue ...)