Lecture 12: Finishing proof of uniqueness (f.g. modules over a PID)

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In the previous lecture we were proving:

<u>Theorem</u>. Let D be a PID, $a_1,\ldots,a_m, b_1,\ldots,b_s \in D\setminus\{0\}$,

$a_1|a_2|\cdots|a_m$, $b_1|b_2|\cdots|b_s$;

If $D^n \oplus \bigoplus_{i=1}^m D/\langle a_i\rangle \simeq D^r \oplus \bigoplus_{i=1}^s D/\langle b_i\rangle$, then $n=r$, $m=s$, and

$\langle a_i\rangle = \langle b_i\rangle$ for $1\le i\le m$.

<u>Recall</u>. Let $M := D^n \oplus \bigoplus_{i=1}^m D/\langle a_i\rangle$. Then

- $\operatorname{rank}(M)=n$ • $\operatorname{Tor}(M) = \bigoplus_{i=1}^m D/\langle a_i\rangle$.
- $\langle a_i\rangle = \langle b_i\rangle \iff \forall\, p$: irreducible, $v_p(a_i) = v_p(b_i)$.
- Let $S_p := D\setminus pD$; then $S_p^{-1}D\, a = S_p^{-1}D\, p^{v_p(a)}$.

Now we focus on $S_p^{-1}M$ and $S_p^{-1}D$. Notice that

$$S_p^{-1}M \simeq (S_p^{-1}D)^n \oplus \bigoplus_{i=1}^m S_p^{-1}D/\langle p^{v_p(a_i)}\rangle,$$ and we need to show

that $v_p(a_i)$'s are uniquely determined base on the structure of

M. So to simplify our notation, we write D instead of $S_p^{-1}D$,

N instead of $\operatorname{Tor}(S_p^{-1}M)$, and $n_i := v_p(a_i)$. So $N = \bigoplus_{i=1}^m D/\langle p^{n_i}\rangle$,

Lecture 12: Uniqueness part of the fundamental theorem of f.g. modules over a PID

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and $n_1 \le n_2 \le \cdots \le n_m$.

Now consider $N \supseteq pN \supseteq p^2 N \supseteq \cdots \supseteq p^{n_m} N = 0$.

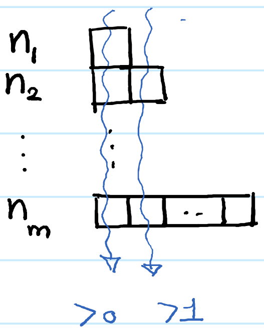
Notice that, for any i, $p^i M / p^{i+1} M$ is an $k(p) := D/\langle p \rangle$ - mod. Since p is irred., $D/\langle p \rangle$ is a field. Hence $p^i M / p^{i+1} M$ is a vector space over $k(p)$.

Notice $p^j M = \bigoplus_{i=1}^{m} p^j \left(D/p^{n_i} D \right) = \bigoplus_{n_i > j} p^j D / p^{n_i} D$

$$\Rightarrow \quad p^j M / p^{j+1} M \simeq \bigoplus_{n_i > j} p^j D / p^{j+1} D \simeq \bigoplus_{n_i > j} k(p).$$

$$\Rightarrow \dim_{k(p)} p^j M / p^{j+1} M = \left| \{ i \in [1..m] \mid n_i > j \} \right|$$

Using the sequence $n_1 \le n_2 \le \cdots \le n_m$, we construct a Young Tableau:



. So the Young Tableau of

$$\dim_{k(p)} M/pM \ge \dim_{k(p)} pM/p^2 M \ge \cdots$$

is the <u>dual</u> of our desired Young Tableau; and so $v_p(a_1) \le \cdots \le v_p(a_m)$ is uniquely determined by M; and this completes proof of uniqueness. ∎

Lecture 12: f.g. modules over a PID

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I did not go over this example in class; please go over it yourself

<u>Ex.</u> Let D be a PID, and $M = D^n \oplus \bigoplus_{i=1}^{m} D/\langle a_i \rangle$ where $a_1 | a_2 | \cdots | a_m$.

Let $d(M) :=$ min # of generators of M. Then

$$d(M) = m+n \geq \operatorname{rank}(M) = n.$$

<u>Pf.</u> Let p be an irreducible factor of a_1. Then

$M/pM \simeq (D/\langle p \rangle)^{m+n}$. Since $D/\langle p \rangle$ is a field, we get

$$d(M/pM) = d\left((D/\langle p \rangle)^{m+n}\right) = \dim_{D/\langle p \rangle} (D/\langle p \rangle)^{m+n} = m+n.$$

$\Rightarrow d(M) \geq d(M/pM) = m+n.$ (I)

On the other hand, M can be generated by $m+n$ elements (why?)

Hence $d(M) \leq m+n$. (II) ; (I), (II) imply the claim. ∎

- Let k be a field, and $A \in M_n(k) = \operatorname{End}(k^n)$. As we have seen earlier k^n can be viewed as a $k[x]$-mod using the following scalar multiplication: $\left(\sum_{i=0}^{m} c_i x^i\right) \cdot v := \sum_{i=0}^{m} c_i A^i v$.

- <u>Claim.</u> k^n is a torsion $k[x]$-mod.

<u>Pf.</u> If not, $\exists\, v \in k^n$ s.t. $\operatorname{ann}(v) = 0$, which implies

$k[x] \hookrightarrow \operatorname{End}(k^n)$; comparing dim. we get a contradiction.

Lecture 12: Rational canonical form

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Claim 2. $\exists!\, f_1(x) \mid f_2(x) \mid \cdots \mid f_m(x)$ (monic poly.) s.t.

$$k^n \simeq k[x]/\langle f_1(x)\rangle \oplus \cdots \oplus k[x]/\langle f_m(x)\rangle$$

as $k[x]$-mod.

Pf. Since $k[x]$ is a PID and k^n is a torsion $k[x]$-mod, we get the above claim using The Fundam. Theor. of f.g. mod. over a PID, (and the corollaries of the proof of uniqueness)

Claim 3. Let $f(x) \in k[x]$ be a polynomial of degree $d > 0$ in $k[x]$.

Then
- $k[x]/\langle f(x)\rangle$ is a k-vector space of dimension d
- $\{\bar{1}, \bar{x}, \ldots, \bar{x}^{d-1}\}$ is a basis of $k[x]/\langle f(x)\rangle$.
- Let $\ell_x : k[x]/\langle f(x)\rangle \to k[x]/\langle f(x)\rangle$ be the multiplication by x map; that means $\ell_x(p(x)+\langle f(x)\rangle) = xp(x)+\langle f(x)\rangle$.

  Then ℓ_x is a k-linear map; and

  the matrix associated with ℓ_x in the basis $\{\bar{1}, \ldots, \bar{x}^{d-1}\}$

  is $C(f) := \begin{bmatrix} 0 & & & -c_0 \\ 1 & & & -c_1 \\ & \ddots & & \vdots \\ & & 1 & -c_{d-1} \end{bmatrix}$ where $f(x) = x^d + c_{d-1}x^{d-1} + \cdots + c_0$.

($C(f)$ is called the companion matrix of f.)

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Pf of claim 3. • $k[x]/\langle f(x)\rangle$ is a ring and $k \hookrightarrow k[x]/\langle f(x)\rangle$ as $\deg f>0$. Hence $k[x]/\langle f(x)\rangle$ is a k-vector space.

• By the long division, $\forall\, g(x)\in k[x]$, $\exists!\; q(x), r(x)\in k[x]$ s.t.

$g(x)=f(x)q(x)+r(x)$ and $\deg r<\deg f$. Hence

$$g(x)+\langle f(x)\rangle = r(x)+\langle f(x)\rangle = c_0'+c_1'x+\cdots+c_{d-1}'x^{d-1}+\langle f(x)\rangle$$

Therefore $(c_0',c_1',\dots,c_{d-1}') \mapsto c_0'+c_1'x+\cdots+c_{d-1}'\,x^{d-1}+\langle f(x)\rangle$

is an isomorphism of k-vector spaces $k^d \to k[x]/\langle f(x)\rangle$,

and the standard basis is mapped to $\mathcal{B}:=\{\bar{1},\bar{x},\dots,\bar{x}^{d-1}\}$.

One can see that ℓ_x is a k-linear map. To find $[\ell_x]_{\mathcal{B}}$,

we have to write $\ell_x(\bar{x}^i)$ as a linear combination of elements of

$\mathcal{B}$: $\quad \bar{1} \xmapsto{\ell_x} \bar{x} \xmapsto{\ell_x} \bar{x}^2 \mapsto \cdots\; \bar{x}^{d-1} \xmapsto{\ell_x} \bar{x}^d = -c_0-c_1\bar{x}-\cdots-c_{d-1}\bar{x}^{d-1}$.

So $[\ell_x]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & 1 & -c_{d-1} \end{bmatrix} = c(f)$.

Claim 4. We can view k^d as a $k[x]$-module using the companion matrix $c(f)$; that means $x\cdot v := c(f)v$. Then $k^d \simeq k[x]/\langle f(x)\rangle$ as $k[x]$-mod.

Lecture 12: The companion matrix

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(Less details were presented in class; hopefully you would find the extended version helpful.)

<u>Pf of claim 4.</u> Let $\phi : k^d \longrightarrow k[x]/\langle f(x)\rangle$,

$$\phi(c_0, \dots, c_{d-1}) := c_0 + c_1 \bar{x} + \dots + c_{d-1} \bar{x}^{d-1}$$

As we discussed in the proof of claim 3, ϕ is an isomorphism of <u>k-vector spaces</u>. Next we check

$$\phi(x \cdot v) = x \bullet \phi(v), \quad (*)$$

which implies the claim.

To see $(*)$, it is enough to check it for the standard basis:

$$x \cdot e_i = c(f)\, e_i = \begin{cases} e_{i+1} & \text{if } i<d \\ -c_0 e_1 - c_1 e_2 - \dots - c_{d-1} e_d & \text{if } i=d \end{cases}$$

$$\Rightarrow \phi(x \cdot e_i) = \begin{cases} \phi(e_{i+1}) & \text{if } i<d \\ -c_0 \phi(e_1) - c_1 \phi(e_2) - \dots - c_{d-1} \phi(e_d) & \text{if } i=d \end{cases}$$

$$= \left\{ \begin{array}{ll} \bar{x}^{i+1} & \text{if } i<d \\ -c_0 - c_1 \bar{x} - \dots - c_{d-1} \bar{x}^{d-1} & \text{if } i=d \end{array} \right\} = \bar{x}^{i+1}.$$

On the other hand, $x \bullet \phi(e_i) = x \bullet \bar{x}^i = \bar{x}^{i+1}$; and the claim follows. ∎