

Lecture 17: More on $\text{Hom}(M, _)$ functor

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We have seen that $F_M: \mathcal{R}\text{-mod} \rightarrow \text{Ab}$ is a left exact functor.

$\mathcal{R}\text{-mod}$ could be either left $\mathcal{R}\text{-mod}$ or right $\mathcal{R}\text{-mod}$.

We say M is an (S, \mathcal{R}) -bi-module if it is a left $S\text{-mod}$ and right $\mathcal{R}\text{-mod}$ and $\forall s \in S, r \in \mathcal{R}, x \in M, (s \cdot x) \cdot r = s \cdot (x \cdot r)$.

One can see that M is an (S, \mathcal{R}) -bi-module exactly when

it is a left $S \times \mathcal{R}^{\text{op}}$ -module. In particular, for commutative

rings \mathcal{R} and S , M is an (S, \mathcal{R}) -bi-module exactly when

it is an $S \times \mathcal{R}$ -mod.

Lemma. Suppose M is an (S, \mathcal{R}) -bi-module. Then for any right $\mathcal{R}\text{-mod}$ N we have that $\text{Hom}_{\mathcal{R}}(M, N)$ is a right

$S\text{-mod}$ with the following scalar multiplication:

$$\forall s \in S, \varphi \in \text{Hom}_{\mathcal{R}}(M, N), x \in M, (\varphi \cdot s)(x) := \varphi(s \cdot x)$$

$$\begin{aligned} \text{Pf. } (\varphi \cdot s)(x_1 r_1 + x_2 r_2) &= \varphi(s \cdot (x_1 r_1 + x_2 r_2)) = \varphi((s \cdot x_1) r_1 + (s \cdot x_2) r_2) \\ &= \varphi(s \cdot x_1) r_1 + \varphi(s \cdot x_2) r_2 = (\varphi \cdot s)(x_1) r_1 + (\varphi \cdot s)(x_2) r_2. \end{aligned}$$

Hence $\varphi \cdot s \in \text{Hom}_{\mathcal{R}}(M, N)$. One can easily check module proper. \blacksquare

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Proposition. Suppose M is an (S, R) -bi-module. Then

$F_M : (\text{right}) R\text{-mod} \rightarrow (\text{right}) S\text{-mod}$ is a well-defined functor.

Pf We have already checked that $F_M(N)$ is a right S -mod.

So we need to check that for $\phi \in \text{Hom}_R(N_1, N_2)$ we have

$$F_M(\phi) \in \text{Hom}_S(F_M(N_1), F_M(N_2)).$$

We have already proved that $F_M(\phi)$ is an abelian group homomorphism. So it is enough to show:

$$\forall \psi \in F_M(N_1), s \in S, (F_M(\phi))(\psi \cdot s) \stackrel{?}{=} (F_M(\phi))(\psi) \cdot s$$

$$F_M(\phi)(\psi \cdot s) = \phi \circ (\psi \cdot s) = F_M(\phi)(\psi) \cdot s$$

$$\begin{aligned} (\phi \circ (\psi \cdot s))(x) &= \phi(\psi(s \cdot x)) = (\phi \circ \psi)(s \cdot x) \\ &= ((\phi \circ \psi) \cdot s)(x) \\ &= ((F_M(\phi))(\psi) \cdot s)(x) \end{aligned}$$

What happens as we compose two functors?

Suppose M is an (S, R) -bi-module and N is a right S -module

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$$\text{Then } \begin{array}{ccccc} \text{(right) } \mathbb{R}\text{-mod} & \xrightarrow{\quad} & \text{(right) } \mathbb{S}\text{-mod} & \xrightarrow{\quad} & \text{Ab} \\ & \downarrow \mathbb{F}_M & & \downarrow \mathbb{F}_N & \\ & \xrightarrow{\quad \mathbb{F}_N \circ \mathbb{F}_M \quad} & & & \end{array}$$

is a functor.

For a right \mathbb{R} -mod L we have

$$\mathbb{F}_N(\mathbb{F}_M(L)) = \text{Hom}_{\mathbb{S}}(N, \text{Hom}_{\mathbb{R}}(M, L)).$$

Q Is $\mathbb{F}_N \circ \mathbb{F}_M : \text{(right) } \mathbb{R}\text{-mod} \rightarrow \text{Ab}$ a representable functor? That means: is there a right \mathbb{R} -module $\mathbb{F}(N, M)$ st.

$\text{Hom}_{\mathbb{S}}(N, \text{Hom}_{\mathbb{R}}(M, L))$ is naturally isomorphic to

$$\text{Hom}_{\mathbb{R}}(\mathbb{F}(N, M), L).$$

The word naturally means: for any L ,

$$\mathbb{F}_N(\mathbb{F}_M(L)) \xrightarrow[\eta_L]{\sim} \mathbb{F}_{\mathbb{F}(N, M)}(L)$$

st. for any $\varphi \in \text{Hom}_{\mathbb{R}}(L_1, L_2)$ we have

$$\begin{array}{ccc} \mathbb{F}_N(\mathbb{F}_M(L_1)) & \xrightarrow[\eta_{L_1}]{\sim} & \mathbb{F}_{\mathbb{F}(N, M)}(L_1) \\ \downarrow \mathbb{F}_N(\mathbb{F}_M(\varphi)) & \circlearrowleft & \downarrow \mathbb{F}_{\mathbb{F}(N, M)}(\varphi) \\ \mathbb{F}_N(\mathbb{F}_M(L_2)) & \xrightarrow[\eta_{L_2}]{\sim} & \mathbb{F}_{\mathbb{F}(N, M)}(L_2) \end{array}$$

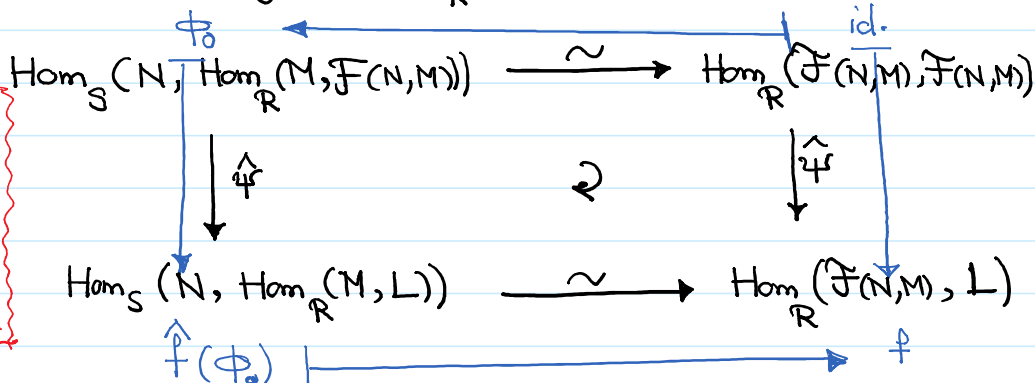
This means η_L 's should be compatible with \mathbb{R} -mod. homomorphisms.

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In particular, for any $f \in \text{Hom}_{\mathbb{R}}(\mathcal{F}(N, M), L)$, we should have

This idea is essentially proof of Yoneda's lemma



Let's take a closer look at elements of $\text{Hom}_S(N, \text{Hom}_{\mathbb{R}}(M, L))$

and $\hat{\varphi}(\phi_0)$.

For $\phi \in \text{Hom}_S(N, \text{Hom}_{\mathbb{R}}(M, L))$, let $f_{\phi}: N \times M \rightarrow L$,

$$f_{\phi}(n, m) := (\phi(n))(m).$$

Then $f_{\phi}(n \cdot s, m) = (\phi(n \cdot s))(m) = (\phi(n) \cdot s)(m) = \phi(n)(s \cdot m)$
 $= f_{\phi}(n, s \cdot m)$ (balanced)

$f_{\phi}(n_1 - n_2, m) = (\phi(n_1 - n_2))(m) = (\phi(n_1))(m) - (\phi(n_2))(m)$
 $= f_{\phi}(n_1, m) - f_{\phi}(n_2, m)$ (linear on N)

$f_{\phi}(n, m_1 r_1 + m_2 r_2) = (\phi(n))(m_1 r_1 + m_2 r_2)$
 $= (\phi(n))(m_1) r_1 + (\phi(n))(m_2) r_2$

(+) $f_{\hat{\varphi}(\phi_0)} = \varphi \circ f_{\phi_0}$ (R-linear on M)
 (the way representable functor is defined.)

Lecture 17: Universal property of tensor product

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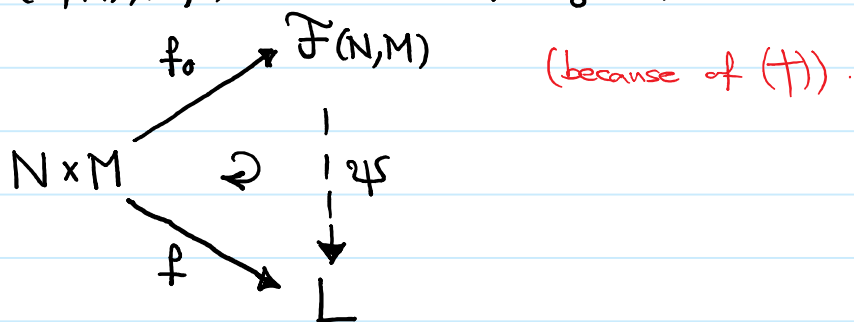
And one can see that, if $f: N \times M \rightarrow L$ is \otimes -balanced, linear in N , R -linear in M , then $(\bigoplus_f(n))(m) := f(n, m)$ defines an element of $\text{Hom}_S(N, \text{Hom}_R(M, L))$. So overall we need to

find a right R -mod $\mathcal{F}(N, M)$ and

$f_0: N \times M \rightarrow \mathcal{F}(N, M)$ with \otimes properties

s.t. if $f: N \times M \rightarrow L$ has \otimes , then

$\exists! \psi \in \text{Hom}_R(\mathcal{F}(N, M), L)$, $\psi \circ f_0 = f$.



So we should define $\mathcal{F}(N, M)$ with least possible relations from

$N \times M$;

free right R -mod.

$$\text{Let } \mathcal{F}(N, M) := \frac{F(N \times M)}{\langle \begin{aligned} &(n \cdot s, m) - (n, s \cdot m), \\ &(n_1 - n_2, m) - (n_1, m) + (n_2, m), \\ &(n, m_1 r_1 + m_2 r_2) - (n, m_1) r_1 - (n, m_2) r_2 \end{aligned} \mid n \in N, m \in M, s \in S, r_i \in R \rangle}$$

and $f_0: N \times M \rightarrow \mathcal{F}(N, M)$, $f_0(n, m) = [(n, m)]$.

Now by the universal property of free mod. and 1st isomorphism theorem we get the above property.