Lecture 17: More on Hom(M,_) functor

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We have seen that $F_M : R\text{-mod} \to Ab$ is a left exact functor.

R-mod could be either left R-mod or right R-mod.

We say M is an (S,R)-bi-module if it is a left S-mod and right R-mod and $\forall s\in S, r\in R, x\in M$, $(s\cdot x)\cdot r = s\cdot(x\cdot r)$.

One can see that M is an (S,R)-bi-module exactly when it is a left $S\times R^{op}$-module. In particular, for commutative rings R and S, M is an (S,R)-bi-module exactly when it is an $S\times R$-mod.

<u>Lemma</u>. Suppose M is an (S,R)-bi-module. Then for any right R-mod N we have that $\mathrm{Hom}_R(M,N)$ is a <u>right S-mod</u> with the following scalar multiplication:

$\forall\ s\in S,\ \psi\in \mathrm{Hom}_R(M,N),\ x\in M,\ (\psi\cdot s)(x) := \psi(s\cdot x)$

<u>Pf</u>. $(\psi\cdot s)(x_1 r_1 + x_2 r_2) = \psi(s\cdot(x_1 r_1 + x_2 r_2)) = \psi((s\cdot x_1) r_1 + (s\cdot x_2) r_2)$

$= \psi(s\cdot x_1) r_1 + \psi(s\cdot x_2) r_2 = (\psi\cdot s)(x_1)\, r_1 + (\psi\cdot s)(x_2)\, r_2.$

Hence $\psi\cdot s \in \mathrm{Hom}_R(M,N)$. One can easily check module proper. ∎

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Proposition. Suppose M is an (S,R)-bi-module. Then

F_M : (right) R-mod $\longrightarrow$ (right) S-mod is a well-defined functor.

Pf We have already checked that $F_M(N)$ is a right S-mod.

So we need to check that for $\phi \in \mathrm{Hom}_R(N_1, N_2)$ we have

$F_M(\phi) \in \mathrm{Hom}_S(F_M(N_1), F_M(N_2))$.

We have already proved that $F_M(\phi)$ is an abelian group homomorphism. So it is enough to show:

$\forall \psi \in F_M(N_1)$, $s \in S$, $\quad (F_M(\phi))(\psi . s) \overset{?}{=} (F_M(\phi))(\psi) . s$

$$F_M(\phi)(\psi.s) = \phi \circ (\psi.s) = F_M(\phi)(\psi) . s$$

$$\boxed{\begin{aligned}(\phi \circ (\psi.s))(x) &= \phi(\psi(s.x)) = (\phi \circ \psi)(s.x) \\ &= ((\phi \circ \psi) . s)(x) \\ &= ((F_M(\phi))(\psi) . s)(x)\end{aligned}}$$

■

What happens as we compose two functors?

Suppose M is an (S,R)-bi-module and N is a right S-module

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Then
$$\begin{tikzcd} (\text{right}) \ R\text{-mod} \arrow[r, "F_M"'] \arrow[rr, bend right, "F_N \circ F_M"'] & (\text{right}) \ S\text{-mod} \arrow[r, "F_N"'] & Ab \end{tikzcd}$$
is a functor.

For a right R-mod L we have
$$F_N(F_M(L)) = \mathrm{Hom}_S(N, \mathrm{Hom}_R(M,L)).$$

$\boxed{Q}$ Is $F_N \circ F_M$: (right) R-mod $\longrightarrow Ab$ a representable functor? That means: is there a right R-module $\mathcal{F}(N,M)$ s.t.

$\mathrm{Hom}_S(N, \mathrm{Hom}_R(M,L))$ is <u>naturally</u> isomorphic to

$\mathrm{Hom}_R(\mathcal{F}(N,M), L)$.

The word <u>naturally</u> means: for any L,
$$F_N(F_M(L)) \xrightarrow[\eta_L]{\sim} F_{\mathcal{F}(N,M)}(L)$$
s.t. for any $\psi \in \mathrm{Hom}_R(L_1, L_2)$ we have
$$\begin{tikzcd} F_N(F_M(L_1)) \arrow[r, "\sim", "\eta_{L_1}"'] \arrow[d, "F_N(F_M(\psi))"] \arrow[dr, phantom, "\circlearrowleft"] & F_{\mathcal{F}(N,M)}(L_1) \arrow[d, "F_{\mathcal{F}(N,M)}(\psi)"] \\ F_N(F_M(L_2)) \arrow[r, "\sim", "\eta_{L_2}"'] & F_{\mathcal{F}(N,M)}(L_2) \end{tikzcd}$$
This means η_L's should be compatible with R-mod. homomorphisms.

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In particular, for any $f \in \mathrm{Hom}_R(\mathcal{F}(N,M), L)$, we should have

$$
\begin{tikzcd}
\Phi_0 & & \mathrm{id}. \arrow[ll, mapsto] \\
\mathrm{Hom}_S(N, \mathrm{Hom}_R(M, \mathcal{F}(N,M))) \arrow[rr, "\sim"] \arrow[d, "\hat{\Psi}"] \arrow[drr, phantom, "\circlearrowright"] & & \mathrm{Hom}_R(\mathcal{F}(N,M), \mathcal{F}(N,M)) \arrow[d, "\hat{\Psi}"] \\
\mathrm{Hom}_S(N, \mathrm{Hom}_R(M, L)) \arrow[rr, "\sim"] & & \mathrm{Hom}_R(\mathcal{F}(N,M), L) \\
\hat{f}(\Phi_0) \arrow[rr, mapsto] & & f
\end{tikzcd}
$$

This idea is essentially proof of Yoneda's lemma

Let's take a closer look at elements of $\mathrm{Hom}_S(N, \mathrm{Hom}_R(M,L))$ and $\hat{\Psi}(\Phi_0)$.

For $\phi \in \mathrm{Hom}_S(N, \mathrm{Hom}_R(M,L))$, let $f_\phi : N \times M \to L$,

$$f_\phi(n,m) := (\phi(n))(m).$$

Then
- $f_\phi(n\cdot s, m) = (\phi(n\cdot s))(m) = (\phi(n)\cdot s)(m) = \phi(n)(s\cdot m)$
  $= f_\phi(n, s\cdot m)$ (balanced)
- $f_\phi(n_1 - n_2, m) = (\phi(n_1 - n_2))(m) = (\phi(n_1))(m) - (\phi(n_2))(m)$
  $= f_\phi(n_1, m) - f_\phi(n_2, m)$ (linear on N)
- $f_\phi(n, m_1 r_1 + m_2 r_2) = (\phi(n))(m_1 r_1 + m_2 r_2)$
  $= (\phi(n))(m_1)\, r_1 + (\phi(n))(m_2)\, r_2$
  (R-linear on M)

(†) $\cdot\ f_{\hat{\Psi}(\Phi_0)} = \Psi \circ f_{\Phi_0}$ (the way representable functor is defined.)

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And one can see that, if $f: N\times M \to L$ is $\circledast$ S-balanced, linear in N, R-linear in M, then $(\Phi_f(n))(m) := f(n,m)$ defines an element of $\mathrm{Hom}_S(N, \mathrm{Hom}_R(M,L))$. So overall we need to find a right R-mod $\mathcal{F}(N,M)$ and

$$f_0 : N\times M \longrightarrow \mathcal{F}(N,M) \text{ with } \circledast \text{ properties}$$

s.t. if $f: N\times M \to L$ has $\circledast$, then

$$\exists!\ \psi \in \mathrm{Hom}_R(\mathcal{F}(N,M), L), \qquad \psi\circ f_0 = f.$$

$$\begin{tikzcd} & \mathcal{F}(N,M) \arrow[d, dashed, "\psi"] \\ N\times M \arrow[ur, "f_0"] \arrow[dr, "f"'] \arrow[r, phantom, "\circlearrowleft"] & {} \\ & L \end{tikzcd}$$

(because of (+)).

So we should define $\mathcal{F}(N,M)$ with least possible relations from $N\times M$;

free right R-mod.

Let $\mathcal{F}(N,M) := F(N\times M) / \langle (n\cdot s, m) - (n, s\cdot m),\ (n_1+n_2, m) - (n_1, m) + (n_2, m),\ (n, m_1 r_1 + m_2 r_2) - (n, m_1) r_1 - (n, m_2) r_2 \mid n\in N, m\in M, s\in S, r_i\in R \rangle$,

and $f_0 : N\times M \to \mathcal{F}(N,M)$, $f_0(n,m) = [(n,m)]$.

Now by the universal property of free mod. and 1^{st} isomorphism theorem we get the above property.