Lecture 19: Tensor functor is right exact

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Theorem. Let M be an (S,R)-bimodule. Then

$$-\otimes_S M : \text{right } S\text{-mod} \to \text{right } R\text{-mod}$$ is a right exact functor.

Pf. Previous lecture we proved that it is a functor. Now we have to show

if $0 \to N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3 \to 0$ is a S.E.S.,

then $N_1 \otimes_S M \xrightarrow{f_1 \otimes id} N_2 \otimes_S M \xrightarrow{f_2 \otimes id} N_3 \otimes_S M \longrightarrow 0$ is exact.

• $\underline{\operatorname{Im}(f_1 \otimes id) \subseteq \ker(f_2 \otimes id.)}$

We know $(f_2 \otimes id) \circ (f_1 \otimes id) = (f_2 \circ f_1) \otimes id = 0$; and the claim follows.

• So we get a right R-module homomorphism

$$\pi : N_2 \otimes_S M / \operatorname{Im}(f_1 \otimes id) \longrightarrow N_3 \otimes_S M, \quad \pi([n_2 \otimes m]) := f_2(n_2) \otimes m.$$

• Let $\psi : N_3 \times M \longrightarrow N_2 \otimes_S M / \operatorname{Im}(f_1 \otimes id.)$, $\psi(n_3, m) := [n_2 \otimes m]$

where $n_2 \in N_2$ and $f_2(n_2) = n_3$.

$\underline{\psi \text{ is well-defined}}$. $f_2(n_2) = f_2(n_2') \Rightarrow n_2 - n_2' \in \ker f_2 = \operatorname{Im} f_1$

$\Rightarrow \exists\, n_1 \in N_1,\ n_2 - n_2' = f_1(n_1)$

$\Rightarrow n_2 \otimes m = n_2' \otimes m + \underbrace{f_1(n_1) \otimes m}_{\operatorname{Im}(f_1 \otimes id)} \Rightarrow [n_2 \otimes m] = [n_2' \otimes m].$

$\underline{S\text{-balanced}}$. $\psi(n_3 \cdot s, m) = [n_2 \cdot s \otimes m] = [n_2 \otimes s.m] = \psi(n_3, s.m)$

$\boxed{f_2(n_2) = n_3 \Rightarrow f_2(n_2 . s) = n_3 . s}$

Lecture 19: Tensor functor is right exact; flat modules

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<u>linear in N_3.</u> $\psi(n_3-n_3', m) = [(n_2-n_2')\otimes m] = [n_2\otimes m] - [n_2'\otimes m]$

$$\boxed{\left.\begin{array}{l} f_2(n_2)=n_3 \\ f_2(n_2')=n_3' \end{array}\right\} \Rightarrow f(n_2-n_2')=n_3-n_3'}$$

$= \psi(n_3, m) - \psi(n_3', m)$

<u>R-linear in M.</u> $\psi(n_3, m_1r_1+m_2r_2) = [n_2\otimes(m_1r_1+m_2r_2)]$

$$= [n_2\otimes m_1]r_1 + [n_2\otimes m_2]r_2 = \psi(n_3,m_1)r_1 + \psi(n_3,m_2)r_2.$$

So $\exists\ \overline{\psi}: N_3\otimes_S M \longrightarrow N_2\otimes_S M/_{\mathrm{Im}(f_1\otimes \mathrm{id})}$, $\overline{\psi}(n_3\otimes m) = [n_2\otimes m]$ where $f_2(n_2)=n_3$.

Therefore π and $\overline{\psi}$ are inverses of each other; this implies

$\mathrm{Im}(f_1\otimes \mathrm{id.}) = \ker(f_2\otimes \mathrm{id.})$ and $\mathrm{Im}(f_2\otimes\mathrm{id.}) = N_3\otimes_S M$; and claim follows. ∎

<u>Corollary</u>. Suppose M is an (S,R)-bimodule. Then $\_\otimes_S M$ is an exact functor if and only if for an injective $f\in \mathrm{Hom}_S(N_1,N_2)$ we get that $f\otimes \mathrm{id}\in \mathrm{Hom}_R(N_1\otimes_S M, N_2\otimes_S M)$ is also injective.

$$0\longrightarrow N_1 \xrightarrow{f_1} N_2 \text{ exact} \Rightarrow 0\longrightarrow N_1\otimes_S M \xrightarrow{f_1\otimes \mathrm{id.}} N_2\otimes_S M \text{ exact}$$

<u>Def.</u> An (S,R)-bimodule is called <u>flat</u> if $\_\otimes_S M$ is an exact functor.

Lecture 19: Associativity of tensor product

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__Remark__. Any left S-mod is an $(S,\mathbb{Z})$-bimodule. So for right S-mod N and left S-mod M, $N\otimes_S M$ is a well-defined abelian group. (And if M is an (S,R)-bimodule, then the two tensor products are "the same".)

And so we can and will talk about flat S-modules.

__Lemma__. Let M be an (S,R)-bimodule, N be a right S-module, and L be a left R-module; then

$$(N\otimes_S M)\otimes_R L \xrightarrow{\simeq} N\otimes_S (M\otimes_R L)$$

$$(n\otimes m)\otimes l \longmapsto n\otimes(m\otimes l)$$

__Pf.__ For $l_0\in L$, let $f_{l_0}: N\times M \longrightarrow N\otimes_S(M\otimes_R L)$,

$$f_{l_0}(n,m) := n\otimes(m\otimes l_0).$$

__S-balanced__. $f_{l_0}(n\cdot s, m) = (n\cdot s)\otimes(m\otimes l_0) = n\otimes s\cdot(m\otimes l_0)$

$$= n\otimes(s.m\otimes l_0) = f_{l_0}(n, s.m)$$

__Linear in N__. $f_{l_0}(n_1-n_2, m) = (n_1-n_2)\otimes(m\otimes l_0)$

$$= n_1\otimes(m\otimes l_0) - n_2\otimes(m\otimes l_0)$$

$$= f_{l_0}(n_1,m) - f_{l_0}(n_2,m)$$

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<u>Linear in M</u>. $f_{l_0}(n, m_1-m_2) = n \otimes ((m_1-m_2)\otimes l_0)$

$$= n\otimes(m_1\otimes l_0) - n\otimes(m_2\otimes l_0)$$

$$= f_{l_0}(n,m_1) - f_{l_0}(n,m_2).$$

So $\exists$ an abelian group homomorphism $\psi_{l_0} : N\otimes_S M \longrightarrow N\otimes_S (M\otimes_R L)$,

$\psi_{l_0}(n\otimes m) = n\otimes(m\otimes l_0)$.

Let $\psi:(N\otimes_S M)\times L \longrightarrow N\otimes_S(M\otimes_R L)$, $\psi(n\otimes m, l) := n\otimes(m\otimes l)$.

We have already proved that ψ is well-defined and linear in the first factor.

<u>R-balanced</u>. $\psi((n\otimes m)\cdot r, l) = \psi(n\otimes mr, l)$

$$= n\otimes(mr\otimes l) = n\otimes(m\otimes rl) = \psi(n\otimes m, rl)$$

<u>Linear in L</u>. $\psi(n\otimes m, l_1 - l_2) = n\otimes(m\otimes l_1 - l_2)$

$$= n\otimes(m\otimes l_1) - n\otimes(m\otimes l_2) = \psi(n\otimes m, l_1) - \psi(n\otimes m, l_2).$$

So $\exists$ an abelian gp homomorphism $\tilde{\psi}: (N\otimes_S M)\otimes_R L \longrightarrow N\otimes_S(M\otimes_R L)$

$$(n\otimes m)\otimes l \longmapsto n\otimes(m\otimes l)$$

Similarly one can prove that $\exists$ an abelian gp hom.

$\tilde{\theta}: N\otimes_S(M\otimes_R L) \longrightarrow (N\otimes_S M)\otimes_R L$, $n\otimes(m\otimes l) \longmapsto (n\otimes m)\otimes l$;

and $\tilde{\theta}$ and $\tilde{\psi}$ are inverse of each other; and the claim follows.

Lecture 19: Tensor product of flat modules

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__Theorem__. Suppose N is a flat right S-mod and M is an (S,R)-bimodule which is a flat right R-mod. Then $N\otimes_S M$ is a flat R-mod.

__Pf.__

$$\begin{tikzcd}
\text{Left } R\text{-mod} \arrow[r, "M\otimes_R -"', "\text{exact functor}"] \arrow[rr, bend right, "N\otimes_S(M\otimes_R -)\ \text{exact functor}"'] & \text{Left } S\text{-mod} \arrow[r, "N\otimes_S -"', "\text{exact functor}"] & Ab
\end{tikzcd}$$

Let $0 \to L_1 \xrightarrow{f_1} L_2 \xrightarrow{f_2} L_3 \to 0$ be a S.E.S.

Then

$$\begin{tikzcd}
 & n\otimes(m\otimes \ell_1) \arrow[r, mapsto] & n\otimes(m\otimes f_1(\ell_1)) & & \\
0 \arrow[r] & N\otimes_S(M\otimes_R L_1) \arrow[r, "id.\otimes(id\otimes f_1)"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & N\otimes_S(M\otimes_R L_2) \arrow[r, "id.\otimes(id\otimes f_2)"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & N\otimes_S(M\otimes_R L_3) \arrow[r] \arrow[d, "\wr"] & 0 \\
 & (N\otimes_S M)\otimes_R L_1 \arrow[r, "id.\otimes f_1"'] & (N\otimes_S M)\otimes L_2 \arrow[r, "id.\otimes f_2"'] & (N\otimes_S M)\otimes_R L_3 & \\
 & (n\otimes m)\otimes \ell_1 \arrow[r, mapsto] & (n\otimes m)\otimes f_1(\ell_1) & &
\end{tikzcd}$$

And so, since the 1^{st} row is a S.E.S, the 2^{nd} row is a S.E.S. as well. ∎