Lecture 21: Tensor product of algebras

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In the previous lecture we mentioned that $A \otimes_R B$ is an $R$-algebra if $A$ and $B$ are $R$-algebras. The following is instrumental in understanding the $R$-algebra structure of many tensor products.

**Proposition.** Suppose $R$ and $S$ are unital commutative rings, and $\phi : S \to R$ is a unital ring homomorphism. Let $I$ be an ideal of the ring $S[x]$ of polynomials over $S$. Then

$$R \otimes_S S[x]/I \cong R[x]/(R \phi(I)).$$

Proof. First we notice that $\phi$ can be extended to a unital ring homomorphism $\phi : S[x] \to R[x]$, $\phi(\sum s_i x^i) := \sum \phi(s_i) x^i$.

- Showing (\(\cong\)) as $R$-modules.

Let $\phi : R \times S[x]/I \to R[x]/(R \phi(I))$,

$\phi(r, p(x) + I) := r \phi(p(x)) + R \phi(I)$.

well-defined. $x \cdot \phi(I) \subseteq \phi(I)$ $\to$ $R \phi(I) \triangleleft R[x]$

$\phi(I)$ additive subgp

- $p(x) + I = p_2(x) + I \Rightarrow p(x) - p_2(x) \in I \Rightarrow r \phi(p_1) - r \phi(p_2) \in R \phi(I)$.
Lecture 21: Tensor product of algebras

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- **S-balanced**: \( f(r \cdot s, p \cdot x + I) = f(r \phi(s), p \cdot x + I) \)
  \[ = r \cdot \phi(s) \cdot \phi(p \cdot x) + R \cdot \phi(I) = r \cdot \phi(s \cdot p \cdot x) + R \cdot \phi(I) \]
  \[ = f(r, s \cdot (p \cdot x + I)) . \]

- **Linearity conditions are easy to check**.

So \( \exists \) an \( R \)-module homomorphism \( \theta: R \otimes_S S[x]/I \rightarrow R[x]/\phi(I) \) such that \( \theta(r \otimes p \cdot x + I) = r \cdot \phi(p \cdot x) + R \cdot \phi(I) \).

Let \( \widetilde{\pi}: R[x] \rightarrow R \otimes_S S[x]/I \), \( \widetilde{\pi}(\sum_{i=0}^{m} r_i \cdot x^i) := \sum_{i=0}^{m} r_i \otimes (x^i + I) \).

**Claim 1.** \( \widetilde{\pi} \) is an \( R \)-mod. homomorphism. \( \checkmark \)

**Claim 2.** \( \phi(I) \subseteq \ker(\widetilde{\pi}) \)

**Proof of claim.** Suppose \( \sum_{i=0}^{m} s_i \cdot x^i \in I \). Then
\[ \widetilde{\pi}(\phi(\sum_{i=0}^{m} s_i \cdot x^i)) = \sum_{i=0}^{m} \phi(s_i) \cdot x^i = \sum_{i=0}^{m} \phi(s_i) \otimes (x^i + I) = \sum_{i=0}^{m} (1 \otimes s_i) \otimes (x^i + I) = \sum_{i=0}^{m} 1 \otimes (s_i \cdot x^i + I) = 0 . \]

Hence \( \exists \pi: R[x]/\phi(I) \rightarrow R \otimes_S S[x]/I \), \( \pi: R \)-mod. hom, and \( \pi(\sum_{i=0}^{m} r_i \cdot x^i + R \cdot \phi(I)) = \sum_{i=0}^{m} r_i \otimes (x^i + I) \).

Therefore \( \theta \) and \( \pi \) are inverse of each other.
So we have that \( \Theta \) is an \( R \)-mod. isomorphism. To show \( \Theta \) is an \( R \)-algebra isomorphism it is enough to show

\[
\Theta((r_1 \otimes \overline{p_1(x)})(r_2 \otimes \overline{p_2(x)})) = \Theta(r_1 \otimes \overline{p_1(x)}) \theta (r_2 \otimes \overline{p_2(x)}) \quad \text{(why?)}
\]

\[
\Theta(r_1 r_2 \otimes \overline{p_1(x)p_2(x)}) = (\overline{r_1 p_1(x) + R \phi(I)})(\overline{r_2 p_2(x) + R \phi(I)})
\]

\[
r_1 r_2 p_1(x)p_2(x) + R \phi(I) = r_1 r_2 p_1(x)p_2(x) + R \phi(I).
\]

**Ex.** \( \mathbb{Q}[x] \otimes \mathbb{Q}[x] \sim \mathbb{Q}[x]/\langle x^2+1 \rangle \)

\[
\sim \mathbb{Q}[x]/\langle x^2+1 \rangle
\]

\[
\sim \mathbb{Q}[x]/\langle (x+i)(x-i) \rangle
\]

\[
\sim \mathbb{Q}[x]/\langle x+i \rangle \oplus \mathbb{Q}[x]/\langle x-i \rangle
\]

\[
\sim \mathbb{Q}[i] \oplus \mathbb{Q}[i].
\]

(think about the reasoning behind each step)

**Ex.** \((\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Z} \mathbb{I}[i] \sim (\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Z} \mathbb{I}[x]/\langle x^2+1 \rangle \)

\[
\sim (\mathbb{Z}/p\mathbb{Z}) [x]/\langle x^2+1 \rangle \quad \text{(*)}
\]

**Case 1.** \( p = 2 \). \( x^2+1 = (x+1)^2 \) in \( \mathbb{Z}/2\mathbb{Z} \). So

\[
\mathbb{Q}[y] \sim (\mathbb{Z}/2\mathbb{Z})[y]/\langle (x+1)^2 \rangle \sim (\mathbb{Z}/2\mathbb{Z})[y]/\langle y^2 \rangle \quad \text{(has nilpotent element.)}
\]
Case 2. \( \exists a_0 \in \mathbb{Z}/p\mathbb{Z} \) s.t. \( a_0^2 + 1 = 0 \). Then \( a_0 \neq -a_0 \) (as \( p \neq 2 \)).

And so \( x^2 + 1 = (x - a_0)(x + a_0) \) and \( \gcd(x - a_0, x + a_0) = 1 \).

Hence \( \left( \mathbb{Z}/p\mathbb{Z} \right)[x]/\langle x^2 + 1 \rangle \cong \left( \mathbb{Z}/p\mathbb{Z} \right)[x]/\langle x - a_0 \rangle \oplus \left( \mathbb{Z}/p\mathbb{Z} \right)[x]/\langle x + a_0 \rangle \cong \left( \mathbb{Z}/p\mathbb{Z} \right) \oplus \left( \mathbb{Z}/p\mathbb{Z} \right) \).

Case 3. \( x^2 + 1 \) has no zero in \( \mathbb{Z}/p\mathbb{Z} \). And so \( x^2 + 1 \) is irreducible in \( \left( \mathbb{Z}/p\mathbb{Z} \right)[x] \). Therefore \( \left( \mathbb{Z}/p\mathbb{Z} \right)[x]/\langle x^2 + 1 \rangle \) is a field of size \( p^2 \).

Example. \( k[y] \otimes k[x] \cong k[y, x] \).
As we have indicated long ago, a lot of algebra has been developed in order to understand zeros of polynomials. And of course the first question is if a given polynomial has a zero:

(Roughly) Suppose \( F \) is a field and \( \phi_{\alpha} \in F[x] \). Can we find a field \( E \) and \( \alpha \in E \) such that \( \phi_{\alpha}(\alpha) = 0 \)?

(More precise) Is there a field extension \( F \subseteq E \) and \( \alpha \in E \) such that \( \phi_{\alpha}(\alpha) = 0 \) where \( \phi_{\alpha} : F[x] \to E \) is the evaluation at \( \alpha \)?

Def. Suppose \( F \subseteq E \) is a field extension. We say \( \alpha \in E \) is algebraic over \( F \) if \( \phi_{\alpha}(\alpha) = 0 \) for some \( \phi_{\alpha} \in F[x] \setminus \{0\} \).

Otherwise we say \( \alpha \) is transcendental over \( F \).

Theorem. Suppose \( F \subseteq E \) is a field extension, and \( \alpha \in E \) is algebraic over \( F \). Then

1. \( \exists ! \) monic polynomial \( m_{\alpha}(x) \in F[x] \) st. \( m_{\alpha}(\alpha) \mid \phi_{\alpha}(\alpha) = 0 \) for \( \phi_{\alpha} \in F[x] \).
2. \( m_{\alpha}(x) \) is irreducible in \( F[x] \).
3. \( F[x] = \left< \sum_{i=0}^{m} a_i x^i \mid a_i \in F \right> \cong F[x]/\langle m_{\alpha}(x) \rangle \); and so \( F[x] \) is a field.
4. \( F\alpha[x] = \left< \sum_{i=0}^{d-1} a_i x^i \mid a_i \in F \right> \) where \( d = \deg m_{\alpha} \); in particular \( \dim_F F\alpha[x] = \deg m_{\alpha} \).