Lecture 21: Tensor product of algebras

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In the previous lecture we mentioned that $A\otimes_R B$ is an R-algebra if A and B are R-algebra. The following is instrumental in understanding the R-algebra structure of many tensor products.

<u>Proposition.</u> Suppose R and S are unital commutative rings, and $\phi: S \to R$ is a unital ring homomorphism. Let I be an ideal of the ring $S[x]$ of polynomials over S. Then

$$R\otimes_S S[x]/I \simeq R[x]/R\phi(I), \qquad (*)$$

$$r\otimes\Big(\big(\sum_{i=0}^m s_i x^i\big)+I\Big) \longmapsto r\sum_{i=0}^m \phi(s_i)x^i + R\phi(I).$$

<u>Pf.</u> <u>First</u> we notice that ϕ can be extended to a unital ring homomorphism $\phi: S[x] \to R[x]$, $\phi(\sum s_i x^i) := \sum \phi(s_i)x^i$.

- <u>Showing $(*)$ as R-modules.</u>

  Let $f: R\times S[x]/I \to R[x]/R\phi(I)$,

  $f(r, p(x)+I) := r\,\phi(p(x)) + R\phi(I)$.

  <u>well-defined</u>. $\left.\begin{array}{l} x\cdot\phi(I) \subseteq \phi(I) \\ \phi(I) \text{ additive subgp} \end{array}\right\} \Rightarrow R\phi(I) \lhd R[x]$

  - $p_1(x)+I = p_2(x)+I \Rightarrow p_1(x)-p_2(x)\in I \Rightarrow r\phi(p_1) - r\phi(p_2) \in R\phi(I)$ ✓

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- S_balanced. $f(r\cdot s\,,\ p(x)+I) = f(r\,\phi(s), p(x)+I)$

$$= r\phi(s)\,\phi(p(x)) + R\phi(I) = r\,\phi(s\,p(x)) + R\phi(I)$$

$$= f(r, s\,(p(x)+I)).$$

- Linearity conditions are easy to check.

So $\exists$ an R-module homomorphism $\theta : R\otimes_S S[x]/I \longrightarrow R[x]/R\phi(I)$,

such that $\theta(r\otimes p(x)+I) = r\,\phi(p(x)) + R\phi(I)$.

. Let $\tilde{\pi} : R[x] \longrightarrow R\otimes_S S[x]/I$, $\tilde{\pi}(\sum_{i=0}^{m} r_i x^i) := \sum_{i=0}^{m} r_i \otimes (x^i + I)$.

Claim 1. $\tilde{\pi}$ is an R-mod. homomorphism. ✓

Claim 2. $\phi(I) \subseteq \ker(\tilde{\pi})$

Pf of claim. Suppose $\sum_{i=0}^{m} s_i x^i \in I$. Then

$$\tilde{\pi}(\phi(\sum_{i=0}^{m} s_i x^i)) = \tilde{\pi}(\sum_{i=0}^{m} \phi(s_i)\, x^i) = \sum_{i=0}^{m} \phi(s_i)\otimes (x^i + I)$$

$$= \sum_{i=0}^{m} (1\cdot s_i)\otimes(x^i + I) = \sum_{i=0}^{m} 1\otimes (s_i x^i + I)$$

$$= 1\otimes (\sum_{i=0}^{m} s_i x^i + I) = 0.$$

Hence $\exists\ \pi : R[x]/R\phi(I) \longrightarrow R\otimes_S S[x]/I$,

π : R-mod. hom; and $\pi(\sum r_i x^i + R\phi(I)) = \sum r_i \otimes (x^i + I)$

Therefore θ and π are inverse of each other.

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So we have that θ is an R-mod. isomorphism. To show θ is an <u>R-algebra</u> isomorphism it is enough to show

$$\theta\big((r_1 \otimes \overline{p_1(x)})(r_2 \otimes \overline{p_2(x)})\big) = \theta(r_1 \otimes \overline{p_1(x)})\, \theta(r_2 \otimes \overline{p_2(x)}) \quad \text{(why?)}$$

$$\theta(r_1 r_2 \otimes \overline{p_1(x) p_2(x)}) \qquad\qquad (r_1 p_1(x) + R\phi(I))(r_2 p_2(x) + R\phi(I))$$

$$r_1 r_2\, p_1(x) p_2(x) + R\phi(I) = r_1 r_2\, p_1(x) p_2(x) + R\phi(I).$$

$\blacksquare$

<u>Ex.</u>

$$\mathbb{Q}[i] \otimes_{\mathbb{Q}} \mathbb{Q}[i] \simeq \mathbb{Q}[i] \otimes_{\mathbb{Q}} \mathbb{Q}[x]/\langle x^2+1\rangle$$

$$\simeq \mathbb{Q}[i][x]/\langle x^2+1\rangle$$

$$\simeq \mathbb{Q}[i][x]/\langle (x+i)(x-i)\rangle$$

$$\simeq \mathbb{Q}[i][x]/\langle x+i\rangle \oplus \mathbb{Q}[i][x]/\langle x-i\rangle$$

$$\simeq \mathbb{Q}[i] \oplus \mathbb{Q}[i].$$

(as $\mathbb{Q}$-algebras).

(think about the reasoning behind each step.)

<u>Ex.</u>

$$(\mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[i] \simeq (\mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[x]/\langle x^2+1\rangle$$

$$\simeq (\mathbb{Z}/p\mathbb{Z})[x]/\langle x^2+1\rangle \qquad (*)$$

<u>Case 1.</u> $p=2$. $\quad x^2+1 = (x+1)^2$ in $(\mathbb{Z}/2\mathbb{Z})[x]$. So

$$(*) \simeq (\mathbb{Z}/2\mathbb{Z})[x]/\langle (x+1)^2\rangle \simeq (\mathbb{Z}/2\mathbb{Z})[y]/\langle y^2\rangle \quad \text{(has nilpotent element.)}$$

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<u>Case 2</u>. $\exists\, a_0 \in \mathbb{Z}/p\mathbb{Z}$ s.t. $a_0^2+1=0$. Then $a_0 \neq -a_0$ (as $p\neq 2$).

And so $x^2+1=(x-a_0)(x+a_0)$ and $\gcd(x-a_0, x+a_0)=1$.

Hence $(\mathbb{Z}/p\mathbb{Z})[x]/\langle x^2+1\rangle \simeq (\mathbb{Z}/p\mathbb{Z})[x]/\langle x-a_0\rangle \oplus (\mathbb{Z}/p\mathbb{Z})[x]/\langle x+a_0\rangle$

$$\simeq (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p\mathbb{Z}).$$

<u>Case 3</u>. x^2+1 has no zero in $\mathbb{Z}/p\mathbb{Z}$. And so x^2+1 is irreducible in $(\mathbb{Z}/p\mathbb{Z})[x]$. Therefore

$$(\mathbb{Z}/p\mathbb{Z})[x]/\langle x^2+1\rangle \text{ is a field of size } p^2.$$

<u>Ex</u>. $k[y] \otimes_k k[x] \simeq k[y,x]$.

Lecture 21: Introduction to field theory

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As we have indicated long ago, a lot of algebra has been developed in order to understand zeros of polynomials. And of course the 1^{st} question is if a given polynomial has a zero:

(Roughly) Suppose F is a field and $p(x)\in F[x]$. Can we find a field E and $\alpha\in E$ such that $p(\alpha)=0$?

(More precise) Is there a field extension $F\subseteq E$ and $\alpha\in E$ such that $\phi_\alpha(p(x))=0$ where $\phi_\alpha: F[x]\longrightarrow E$ is the evaluation at α?

<u>Def.</u> Suppose $F\subseteq E$ is a field extension. We say $\alpha\in E$ is <u>algebraic over F</u> if $p(\alpha)=0$ for some $p(x)\in F[x]\setminus\{0\}$.
Otherwise we say α is <u>transcendental over F</u>.

<u>Theorem.</u> Suppose $F\subseteq E$ is a field extension, and $\alpha\in E$ is algebraic over F. Then

(1) $\exists!$ monic polynomial $m_\alpha(x)\in F[x]$ s.t. $m_\alpha(x)\mid p(x) \Leftrightarrow p(\alpha)=0$ for $p(x)\in F[x]$.

(2) $m_\alpha(x)$ is irreducible in $F[x]$.

(3) $F[\alpha]:=\{\sum_{i=0}^{m} a_i\alpha^i \mid a_i\in F\}\simeq F[x]/\langle m_\alpha(x)\rangle$; and so $F[\alpha]$ is a field.

(4) $F[\alpha]=\{\sum_{i=0}^{d_0-1} a_i\alpha^i \mid a_i\in F\}$ where $d_0=\deg m_\alpha$; in particular $\dim_F F[\alpha]=\deg m_\alpha$.