We were proving the following theorem:

**Theorem.** Let $F$ be a field. Then there is a field extension $E/F$ such that $E$ is algebraically closed.

**Pf.** Let $A := F[x_f]_{f: \text{monic}, f \in \text{Fns}(F)}$ and $I = \langle f(x_f) | f: \text{monic}, f \in \text{Fns}(F) \rangle$.

**Claim.** $I$ is a proper ideal.

**Pf.** If not, $1 \in I$. So $\exists g_1, \ldots, g_m \in A$ and $f_1, \ldots, f_m$ s.t.

$$g_1(x_{f_1}) f_1(x_{f_1}) + \ldots + g_m(x_{f_m}) f_m(x_{f_m}) = 1.$$

To make symbols more clear, let $y_i = x_{f_i}$ and $y_{m+1}, \ldots, y_n$ be the rest of variables involved in $g_i$’s. So

$$g_1(y_1, \ldots, y_n) f_1(y_1) + \ldots + g_m(y_1, \ldots, y_n) f_m(y_m) = 1. \quad (\ast)$$

Let $E'$ be the splitting field of $f_1(t) f_2(t) \ldots f_m(t)$ over $F$.

And let $\alpha_1, \ldots, \alpha_m \in E'$ s.t. $f_1(\alpha_1) = f_2(\alpha_2) = \ldots = f_m(\alpha_m) = 0$.

Let’s evaluate $\ast$ at $(\alpha_1, \ldots, \alpha_m, 0, \ldots, 0)$; and we get $0 = 1$,

which is a contradiction.

Let $\mathfrak{m}$ be a maximal ideal of $A$ s.t. $\mathfrak{m} \supseteq I$. Let $E_1 := A/\mathfrak{m}$.

So $E_1$ is a field; and $F \cap \mathfrak{m} = 0$ implies $F \hookrightarrow E_1$. 


Lecture 24: Tower of algebraic extensions

Thursday, March 1, 2018 8:36 PM

Claim. Any monic polynomial $f(x) \in F[x] \setminus F$ has a zero in $E_1$.

Pf. Let $\alpha_1 := x_1 + \theta \in E_1$. Then $f(\alpha_1) = f(x_1) + \theta = \theta \in \theta \setminus \theta$.

We do the same construction again and again to get a tower of field extensions: $F \subseteq E_1 \subseteq E_2 \subseteq \ldots$

Let $E := \bigcup_{i=1}^{\infty} E_i$.

Claim. $E$ is a field.

Pf. $\alpha \in E$, $\beta \in E \setminus \theta \Rightarrow \exists i \text{ st. } \alpha \in E_i$ and $\beta \in E_i \setminus \theta$.

So $\alpha \pm \beta$, $\alpha \beta^{\pm 1} \in E_i \Rightarrow \alpha \pm \beta$, $\alpha \beta^{\pm 1} \in E$.

Claim. $E$ is algebraically closed.

Pf. Let $f(x) := \sum a_i x^i \in E[x]$. Then $\exists j \text{ st. } f(x) \in E_j[x]$.

$\Rightarrow f(x)$ has a zero in $E_{j+1} = f(x)$ has a zero in $E$.

Proposition. Suppose $E/F$ and $K/E$ are algebraic extensions. Then $K/F$ is an algebraic extension.

Pf. Let $\alpha \in K$. Then $\alpha^n + e_{n-1} \alpha^{n-1} + \ldots + e_0 = 0$ for some $e_i$'s $\in E$.

Since $e_i$'s are algebraic over $F$, $[F(e_0, \ldots, e_{n-1} : F) < \infty$.

Since $\alpha$ is algebraic over $F(e_0, \ldots, e_{n-1})$,

$[F(e_0, \ldots, e_{n-1}, \alpha : F(e_0, \ldots, e_{n-1})] < \infty$. Hence $F(e_0, \ldots, e_{n-1}, \alpha) \subseteq F$ is a finite extension. Therefore $\alpha$ is algebraic over $F$.  \qed
Theorem. Let $F$ be a field. Then there is an algebraically closed field extension $E/F$ such that $E$ is algebraically closed.

Proof. Let $\tilde{E}/F$ be a field extension such that $\tilde{E}$ is algebraically closed (there is such a field by the previous theorem). Let $E$ be the algebraic closure of $F$ in $\tilde{E}$. So $E/F$ is an algebraic field extension.

Claim $E$ is algebraically closed.

Proof. Let $f(\alpha) \in E[x]$. Since $\tilde{E}$ is algebraically closed, $\exists \alpha \in \tilde{E}$ s.t. $f(\alpha) = 0$. So $E[\alpha]/F$ is algebraic. Since $E/F$ is algebraic, by the previous proposition we deduce that $E[\alpha]/F$ is algebraic. Hence $\alpha$ is algebraic over $F$; this implies $\alpha \in E$; and claim follows.

Definition. We call $E$ an algebraic closure of $F$ if $E/F$ is algebraic and $E$ is algebraically closed.

So far we have proved the existence of an algebraic closure. Next we show it is unique up to isomorphism.
Theorem. (1) Let $F$ be a field and $\Omega$ be an algebraically closed field. Suppose $E$ is the splitting field of $f \in \text{Fix}\Omega \setminus F$ over $F$; and $\sigma : F \to \Omega$. Then $\exists \bar{\sigma} : E \cong \Omega$ s.t. $\bar{\sigma}|_F = \sigma$.

(2) Let $E$ and $E'$ be two algebraic closures of $F$, and $\sigma' : F \to E'$. Then $\exists \phi : E \cong E'$ s.t. $\phi|_F = \sigma'$.

**Proof.** (1) Since $\Omega$ is algebraically closed, $\sigma(f)(x) = (x-\omega_1) \cdots (x-\omega_n)$ for some $\omega_1, \ldots, \omega_n$. Then $E' := \sigma(F)[\omega_1, \ldots, \omega_n]$ is the splitting field of $\sigma(f)$ over $\sigma(F)$. Hence $\exists \bar{\sigma} : E \cong E \subseteq \Omega$ s.t. $\bar{\sigma}|_F = \sigma$.

(2) Let $\Sigma := \{(K, \sigma) \mid F \subseteq K \subseteq E, \sigma : K \to E', \text{subfield}\sigma|_F = \sigma\}$. We say $(K_1, \sigma_1) \preceq (K_2, \sigma_2)$ if and only if $K_1 \subseteq K_2$ and $\sigma_2|_{K_1} = \sigma_1$. $(\Sigma, \preceq)$ is a POSet.

**Claim.** $\Sigma$ has a maximal element.

**Proof.** By Zorn’s lemma, it is enough to show any chain has an upper bound. Suppose $\{K_i, \sigma_i\}_{i \in I}$ is a chain. Let $K := \bigcup_{i \in I} K_i$ and
\( \sigma : K \to E', \quad \sigma'(a) = \sigma_i(a) \) if \( a \in K_i \). Notice that \( \sigma' \) is well-defined.

if \( a \in K_i \) and \( a \in K_j \), as \( \mathcal{S}(K_i, \sigma_i) \) is a chain, w.l.o.g.
we can and will assume \( (K_i, \sigma_i) \triangleleft (K_j, \sigma_j) \). Hence \( K_i \subseteq K_j \)
and \( \sigma_j |_{K_i} = \sigma_i \). And so \( \sigma_j(a) = \sigma_i(a) \).

Show that \( \bigcup_{i \in I} K_i \) is a field.

\( \forall a \in F \subseteq K_i \) (\( \forall i \in I \)), \( \sigma'(a) = \sigma_i(a) = a \).

Hence \( (K, \sigma') \in \Sigma \) and \( (K_i, \sigma_i) \triangleleft (K, \sigma) \) for any \( i \in I \).

Therefore \( (\Sigma, \triangleleft) \) has a maximal element. Suppose \( (K, \sigma') \) is a maximal element of \( \Sigma \).

Claim. \( K = E \).

(We will prove this in the next lecture.)