Homework 1

Saturday, January 12, 2019 2:36 AM

1. Prove that the following polynomials are irreducible; p is prime.

(a) $x^{p-1} + y^2 x^{p-2} + y^2 x^{p-3} + \cdots + y^2$ in $\mathbb{Q}[x,y]$

(b) $1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^p}{p!}$ in $\mathbb{Q}[x]$

(c) $x^n - y$ in $F[x,y]$.

(d) x^2+y^2-2 in $F[x,y]$ where $\mathrm{char}(F) \neq 2$.

(e) $x^4 + 12x^3 - 9x + 6$ in $\mathbb{Q}[i][x]$.

2. Prove that $x^{p^n} - x + a$ does not have a zero in $\mathbb{Q}$ if p is prime, $a \in \mathbb{Z}$, and $p \nmid a$.

3. (a) Prove that in $(\mathbb{Z}/p\mathbb{Z})[x]$ we have

$x(x-1)\cdots(x-(p-1)) = x^p - x$, where p is prime.

(b) Deduce that $(p-1)! \equiv -1 \pmod{p}$.

4. Suppose A is a unital commutative ring, and $\mathfrak{p} \in \operatorname{Spec} A$.

(a) Prove that $A_{\mathfrak{p}}^{\times} = A_{\mathfrak{p}} \setminus \mathfrak{p}A_{\mathfrak{p}}$ where

$\mathfrak{p}A_{\mathfrak{p}} = \{ \frac{a}{s} \mid a \in \mathfrak{p},\ s \in A \setminus \mathfrak{p} \}$.

(b) Prove that $\mathrm{Max}(A_{\mathfrak{p}}) = \{ \mathfrak{p}A_{\mathfrak{p}} \}$.

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5. Let k be a field. For a subset A of the ring $k[x_1,\dots,x_n]$ of polynomials. Let $V(A) := \{\vec{v} \in k^n \mid \forall f(x_1,\dots,x_n) \in A, f(\vec{v}) = 0\}$. And, for a subset X of k^n, let

$$I(X) := \{f \in k[x_1,\dots,x_n] \mid \forall \vec{v} \in X,\ f(\vec{v}) = 0\}.$$

(a) Prove that $I(X) \lhd k[x_1,\dots,x_n]$.

(b) Prove that, $\forall \emptyset \neq A \subseteq k[x_1,\dots,x_n], V(I(V(A))) = V(A)$.

(c) Prove that for any $\emptyset \neq A \subseteq k[x_1,\dots,x_n]$ there are finitely many polynomials $f_1,\dots,f_m$ s.t.

$$V(A) = V(f_1, f_2, \dots, f_m).$$

(d) Suppose $I \lhd k[x_1,\dots,x_n]$. Prove that $\sqrt{I} \subseteq I(V(I))$, where $\sqrt{I} := \{f \in k[x_1,\dots,x_n] \mid \exists m, f^m \in I\}$.