Homework 2

Saturday, January 12, 2019 2:45 AM

$\boxed{1}$ (Opposite ring) (a) Let R be a unital ring. Prove that

$\mathrm{End}_R(R) \simeq R^{op}$ as rings.

(b) Suppose $\exists\, \tau : R \to R$ s.t. • $\tau(x+y) = \tau(x) + \tau(y)$
• $\tau(xy) = \tau(y)\tau(x)$
• $\tau(\tau(x)) = x$.

Prove that $R \simeq R^{op}$.

(c) Prove that $M_n(R)^{op} \simeq M_n(R^{op})$; in particular

$M_n(R)^{op} \simeq M_n(R)$ if R is commutative.

(d) Suppose G is a group and $\mathbb{C}G$ is the group ring of

G over $\mathbb{C}$. Prove that $\mathbb{C}G^{op} \simeq \mathbb{C}G$.

(e) [NOT part of HW assignment] Can you find a ring R such

that $R \not\simeq R^{op}$?

$\boxed{2}$ (Torsion submod.) Suppose R is an integral domain and M is

an R-mod. Let $\mathrm{Tor}(M) := \{ m \in M \mid \exists\, r \in R \setminus \{0\},\ rm = 0 \}$.

(a) Prove that $\mathrm{Tor}(M)$ is a submod. of M.

(b) $\mathrm{Tor}(M/\mathrm{Tor}(M)) = 0$. ($M/\mathrm{Tor}(M)$ is torsion-free.)

Homework 2

Friday, January 26, 2018 10:45 AM

$\boxed{3.}$ (Annihilator) Let R be a unital ring and M be an R-mod.

The annihilator $\mathrm{Ann}(M)$ of M is

$$\mathrm{Ann}(M) := \{r \in R \mid \forall m \in M,\ r.m = 0\}.$$

And the annihilator $\mathrm{Ann}(m)$ of an element m of M is

$$\mathrm{Ann}(m) := \{r \in R \mid r.m = 0\}.$$

(Notice that $\mathrm{Ann}(M) = \bigcap_{m \in M} \mathrm{Ann}(m)$.)

(a) Prove that $\mathrm{Ann}(m)$ is a left ideal.

(b) Give an example where $\mathrm{Ann}(m)$ is NOT a two-sided ideal. (Hint. For instance think about $M_n(\mathbb{C})$; you have seen before that ideals of $M_n(R)$ are of the form $M_n(I)$ where $I \trianglelefteq R$; in particular $M_n(D)$ does not have a non-trivial two sided ideal if D is a division ring.)

(c) Prove that $\mathrm{Ann}(M)$ is a (both sided) ideal of R.

(d) Find a $\mathbb{Z}$-mod. M s.t. $\mathrm{Ann}(M) = 0$ and $\forall m \in M$, $\mathrm{Ann}(m) \neq 0$.

Homework 2

Friday, January 26, 2018 1:23 PM

(<u>Remark</u>. We say M is a faithful R-mod if $\mathrm{Ann}(M)=0$.)

(e) An R-mod M is an $R/\mathrm{Ann}(M)$-mod w.r.t.

$$(r+\mathrm{Ann}(M))\cdot x := r\cdot x \quad \text{scalar multiplication.}$$

$\boxed{4.}$ Let R be a unital ring, $I \trianglelefteq R$, and M be an R-mod.

Let $IM := \{\sum_{i=1}^{m} r_i \cdot x_i \mid r_i \in I,\ x_i \in M\}$.

(a) Prove that IM is a submodule of M.

(b) Prove that $I \subseteq \mathrm{Ann}(M/IM)$; and deduce that

M/IM can be viewed as an R/I-mod via

$$(r+I)\cdot(x+IM) := rx + IM.$$

$\boxed{5.}$ Let $V = \bigoplus_{i=1}^{\infty} \mathbb{C} v_i$ be a countable dimensional vector space over $\mathbb{C}$. Let $R := \mathrm{End}_{\mathbb{C}}(V)$. Prove that as R-modules

$R \simeq R \oplus R$.

(<u>Hint</u>. Use "projection" to odd and even components (or any other partition to two infinite sets.))

$\boxed{6.}$ Let G be a group, and M be an abelian group. Give an

Homework 2

Friday, January 26, 2018 1:33 PM

explicit bijection between the set of linear G-actions on M and $\mathbb{Z}G$-module structures on M.

(You can use without proof that a linear G-action on M is given by $\mathrm{Hom}(G, \mathrm{Aut}(M))$ (group homomorphisms).)

$\boxed{\text{Reading before problem}}$ Determinant can be defined for matrices with entries in a unital commutative ring:

$$\det[a_{ij}] := \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$ where

S_n is the symmetric group, and $\mathrm{sgn}: S_n \to \{\pm 1\}$ is the sign group homomorphism. Similar to the $n\times n$ matrices over a field, one can define minors of $x = [a_{ij}]$.

The l,k-minor of $x = [a_{ij}]$ is the determinant of the $(n-1)\times(n-1)$ matrix $x(l,k)$ that one gets after removing the l^{th} row and the k^{th} column.

$$x(l,k) := \begin{bmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{l1} & \cdots & a_{lk} & \cdots & a_{ln} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nk} & \cdots & a_{nn} \end{bmatrix}$$

Similar to Cramer's rule, we can define the adjunct matrix

Homework 2

Saturday, January 19, 2019 10:30 AM

$\mathrm{adj}(x)$ of x. The (i,j)-entry of $\mathrm{adj}(x)$ is $(-1)^{i+j}\det x(j,i)$.

Here are the main properties of $\det: M_n(A) \to A$.

(1) det is multi-linear with respect to columns.

(1′) det is multi-linear with respect to rows.

(2) $\det(I) = 1$.

(3) If x has two identical rows, then $\det x = 0$

(3′) If x has two identical columns, then $\det x = 0$

(4) $\mathrm{adj}(x) \cdot x = x \cdot \mathrm{adj}(x) = \det(x)\, I$.

(5) $\forall\, x, y \in M_n(A)$, $\det(xy) = \det(x)\det(y)$.

$\boxed{7}$ (a) Suppose A is a unital commutative ring, and $GL_n(A) := M_n(A)^{\times}$.

Prove that $x \in GL_n(A) \iff \det x \in A^{\times}$.

(b) Suppose A is a unital commutative ring and $\mathrm{Max}(A) = \{\mathfrak{m}\}$.

Suppose $\phi: A^n \to A^n$ is an A-mod. homomorphism and let

$x_\phi \in M_n(A)$ be its associated matrix. Convince yourself that

ϕ is a bijection if and only if $x_\phi \in GL_n(A)$.

Prove the following statements are equivalent:

Homework 2

Saturday, January 19, 2019 10:33 AM

(1) $\phi: A^n \to A^n$ is surjective.

(2) $\overline{\phi}: (A/\mathfrak{m})^n \to (A/\mathfrak{m})^n$ is bijective, where $\overline{\phi}$ is induced by ϕ.

(3) $\phi: A^n \to A^n$ is bijective.

(Hint. Show (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3). Use linear algebra to show $\det(\phi) \notin \mathfrak{m}$.)

(c) Suppose A is a unital commutative ring, and $\phi: A^n \to A^n$ is an A-mod. homomorphism. Prove that

$$\phi \text{ is surjective} \Leftrightarrow \phi \text{ is bijective.}$$

(Hint. Use Problem 1.c, 1.d, 3.b)