Math200b, homework 4

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Direct sum vs direct product.

1. Suppose $\{M_i\}_{i\in I}$ is a family of A-modules and N is an A-module. Prove that

   (a) $\mathrm{Hom}_A(\oplus_{i\in I} M_i, N) \simeq \prod_{i\in I} \mathrm{Hom}_A(M_i, N)$,

   (b) $\mathrm{Hom}_A(N, \prod_{i\in I} M_i) \simeq \prod_{i\in I} \mathrm{Hom}_A(N, M_i)$

   as abelian groups.

2. (a) Let $\phi \in \mathrm{Hom}(\prod_{i=1}^{\infty} \mathbb{Z}, \mathbb{Z})$; let $e_j \in \prod_{i=1}^{\infty} \mathbb{Z}$ be

$$e_j(i) := 0 \text{ if } i \neq j \text{ and } e_i(i) = 1.$$

Suppose $\phi(e_j) = n_j \neq 0$ for any j. Choose a sequence of positive integers $1 =: k_1 < k_2 < \cdots$ such that

$$k_{j+1} \nmid k_j! n_j. \tag{1}$$

Consider

$$\Sigma := \{(a_i)_{i=1}^{\infty} | \, a_i \in \{0, k_i!\}\}. \tag{2}$$

(a-1) Argue why there are two distinct elements $(a_i)_{i=1}^{\infty}$ and $(a_i')_{i=1}^{\infty}$ of Σ such that

$$\phi((a_i)_{i=1}^{\infty}) = \phi((a_i')_{i=1}^{\infty}). \tag{3}$$

(a-2) Suppose i_0 is the first index where $a_{i_0} \neq a_{i_0}'$. Show that

$$\phi((a_{i_0} - a_{i_0}')e_{i_0}) \notin k_{i_0+1}\mathbb{Z}, \qquad \text{(Hint: use (1))}$$

and

$$\phi((a_{i_0} - a_{i_0}')e_{i_0}) \in k_{i_0+1}\mathbb{Z}; \qquad \text{(Hint: use (2) and (3))}$$

and get a contradiction.

(b) Use part (a) to deduce

$$\mathrm{Hom}(\prod_{i=1}^{\infty}\mathbb{Z},\mathbb{Z}) \to \bigoplus_{i=1}^{\infty}\mathbb{Z},$$
$$\phi \mapsto (\phi(e_i))_{i=1}^{\infty}$$

is an isomorphism. (**Hint**: suppose $\oplus_{i=1}^{\infty}\mathbb{Z} \subseteq \ker\phi$; then show $p^n | \phi(pa_1, p^2a_2, p^3a_3, \ldots)$ for any n and deduce that $\phi(pa_1, p^2a_2, p^3a_3, \ldots) = 0$; observe that any element $(b_1, b_2, \ldots)$ can be written as a sum of two elements of the form $(2a_1, 2^2a_2, \ldots)$ and $(3a_1, 3^2a_2, \ldots)$.)

(c) Use part (b) to show $\prod_{i=1}^{\infty}\mathbb{Z}$ is not a free abelian group.

(d) Use part (b) to show

$$\mathrm{Hom}(\prod_{i=1}^{\infty}\mathbb{Z}/\bigoplus_{i=1}^{\infty}\mathbb{Z},\mathbb{Z}) = 0.$$

Towards Artin-Wedderburn's theorem.

Suppose M is a simple A-module and let $D := \mathrm{End}_A(M)$.

1. Prove that $\mathrm{End}_A(M^n) \simeq M_n(D)$ as rings.

2. Suppose M_i's are simple A-modules, and $M_i \not\simeq M_j$ as A-modules.

   (a) For $\phi \in \mathrm{End}_A(\oplus_{i=1}^m M_i^{n_i})$, prove that

   $$\phi(M_i^{n_i}) \subseteq M_i^{n_i}.$$

   (b) Prove that

   $$\mathrm{End}_A(\oplus_{i=1}^m M_i^{n_i}) \simeq \mathrm{M}_{n_1}(D_1) \oplus \cdots \oplus \mathrm{M}_{n_m}(D_m)$$

   as rings where $D_i := \mathrm{End}_A(M_i)$.

3. Suppose $A \simeq M_1^{n_1} \oplus \cdots \oplus M_m^{n_m}$ as A-modules, where M_i's are simple A-modules and $M_i \not\simeq M_j$. Prove that

   $$A \simeq \mathrm{M}_{n_1}(D_1^{op}) \oplus \cdots \oplus \mathrm{M}_{n_m}(D_m^{op})$$

   where $D_i = \mathrm{End}_A(M_i)$ are division rings.

Remark. Using problem 5 of the first homework set of math200a, you can show that the group ring $\mathbb{C}G$ of a finite group G is isomorphic to $M_1^{n_1} \oplus \cdots \oplus M_m^{n_m}$ as a $\mathbb{C}G$-module. And so by the above problem after showing $D_i = \mathbb{C}$, you get

$$\mathbb{C}G \simeq \mathrm{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathrm{M}_{n_m}(\mathbb{C});$$

this gives us a lot of information on irreducible representations of G. (It is the starting point of representation theory of finite groups.)

Nilpotent matrices.

1. Suppose k is a field and N_1 and N_2 are two nilpotent matrices in $M_n(k)$. Prove that N_1 and N_2 are similar if and only if $\dim_k \ker(N_1^j) = \dim_k \ker(N_2^j)$ for any $j \in [1..n]$.

2. Suppose A is a reduced unital commutative ring; that means $\mathrm{Nil}(A) = 0$ (A has no non-zero nilpotent element). Suppose $N \in M_n(A)$ is a nilpotent matrix. Prove that $N^n = 0$.

   (Hint: the same statement for fields $\Rightarrow$ for integral domains $\Rightarrow$ for $A/\mathfrak{p}$ where $\mathfrak{p} \in \mathrm{Spec}(A) \Rightarrow$ the general case.)

Diagonalizable matrices.

Suppose k is a field, $A \in M_n(k)$, and the characteristic polynomial $f_A(x) = \prod_{i=1}^{m}(x - \lambda_i)^{k_i}$ where $\lambda_i \in k$ and $\lambda_i \neq \lambda_j$ if $i \neq j$.

1. Suppose A is diagonalizable over k; that means for some $g \in \mathrm{GL}_n(k)$, gAg^{-1} is a diagonal matrix. Prove that $m_A(x) = \prod_{i=1}^m (x - \lambda_i)$ where $m_A(x)$ is the minimal polynomial of A.

2. Prove that A is diagonalizable over k if and only if the minimal polynomial $m_A(x)$ of A has distinct zeros.

3. Suppose $A_1, \ldots, A_l \in \mathrm{M}_n(k)$ are diagonalizable and pairwise commuting; that means $A_i A_j = A_j A_i$ for any i, j. Prove that A_i's are simultaneously diagonalizable; that means there is $g \in \mathrm{GL}_n(k)$ such that gA_ig^{-1} is diagonalizable for any i.

   (Hint: Suppose λ_i's are distinct eigenvalues of A_1. Show

   $$k^n = \oplus_{i=1}^m \ker(A - \lambda_i I), \ \ A_j(\ker(A - \lambda_i I)) \subseteq \ker(A - \lambda_i I);$$

   and prove the claim by induction on l.)

4. Suppose $\{A_i\}_{i \in I}$ is a family of pairwise commuting diagonalizable elements of $\mathrm{M}_n(k)$ where k is a field. Prove that A_i's are simultaneously diagonalizable.

   (Hint: Consider the k-span of $\{A_i\}_{i \in I}$.)

Noetherian and a finite cover of $\mathrm{Spec}(A)$.

Suppose A is a unital commutative ring. For $f \in A$, let $\mathcal{O}_f := \{\mathfrak{p} \in \mathrm{Spec}(A) | f \notin \mathfrak{p}\}$ and $A_f := S_f^{-1}A$ where $S_f := \{1, f, f^2, \ldots\}$.

1. Show that for $f_i \in A$ and $n \in \mathbb{Z}^+$, we have $\mathcal{O}_{f_i^n} = \mathcal{O}_{f_i}$ and
$$\bigcup_{i=1}^{m} \mathcal{O}_{f_i} = \mathrm{Spec}(A) \Leftrightarrow \langle f_1, \ldots, f_m \rangle = A.$$

2. Suppose $\bigcup_{i=1}^{m} \mathcal{O}_{f_i} = \mathrm{Spec}(A)$. Suppose M is an A-module, and $N \subseteq M$ is a submodule. Suppose $S_{f_i}^{-1}N = S_{f_i}^{-1}M$ for any i. Prove that $N = M$.

   (Hint: For $x \in M$, consider $\{a \in A | ax \in N\}$.)

3. Suppose $\bigcup_{i=1}^{m} \mathcal{O}_{f_i} = \mathrm{Spec}(A)$. Suppose M is an A-module, and $S_{f_i}^{-1}M$ is a finitely generated A_{f_i}-module for any i. Prove that M is a finitely generated A-module.

   (Hint: Use the previous problem.)

4. Suppose $\bigcup_{i=1}^{m} \mathcal{O}_{f_i} = \mathrm{Spec}(A)$, and A_{f_i}'s are Noetherian. Prove that A is Noetherian.

   (Hint: Use the previous problem for $\mathfrak{a} \trianglelefteq A$)

(Remark. Based on the previous homework assignment, you can see that $\mathcal{O}_f \longrightarrow \mathrm{Spec}(A_f), \mathfrak{p} \mapsto S_f^{-1}\mathfrak{p}$ is a bijection. So we are more or less saying that having a Noetherian (affine) finite cover of $\mathrm{Spec}(A)$ implies that A is Noetherian.)

Projective module.

1. Suppose P and P' are projective A-modules, and

$$0 \longrightarrow K \longrightarrow P \xrightarrow{f} M \longrightarrow 0$$

and

$$0 \longrightarrow K' \longrightarrow P' \xrightarrow{f'} M \longrightarrow 0$$

are short exact sequences of A-modules. Prove that

$$P \oplus K' \simeq P' \oplus K.$$

Hint: Let $L := \{(x, x') \in P \oplus P' | \, f(x) = f'(x')\}$. Show that L is a submodule of $P \oplus P'$. Notice that the following diagram is commuting and each row and column is an exact sequence; and then use the assumption that P and

P' are projective to deduce $L \simeq P \oplus K'$ and $L \simeq P' \oplus K$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \ker \pi' & \xrightarrow{\sim} & K' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker \pi & \longrightarrow & L & \xrightarrow{\pi'} & P' & \longrightarrow & 0 \\
 & & \downarrow \wr & & \downarrow \pi & & \downarrow f' & & \\
0 & \longrightarrow & K & \longrightarrow & P & \xrightarrow{f} & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

2. Suppose $(A, \mathfrak{m})$ is a <u>local</u> unital commutative ring; that means $\mathrm{Max}(A) = \{\mathfrak{m}\}$.

   (a) (Nakayama's lemma) Suppose M is a finitely generated A-module. Suppose $M = \mathfrak{m}M$ where

   $$\mathfrak{m}M = \{\sum_{i=1}^{n} a_i x_i \mid a_i \in \mathfrak{m}, x_i \in M\}.$$

   Prove that $M = 0$.

(Hint: Let $y_1, \ldots, y_d$ be a generating set of M. By assumption, $\exists a_{ij} \in \mathfrak{m}$ such that

$$y_i = \sum_{j=1}^{d} a_{ij} y_j.$$

Hence $(I - [a_{ij}])\begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix} = 0$. Show that $I - [a_{ij}] \in \mathrm{GL}_d(A)$; and deduce $y_i = 0$; and so $M = 0$.)

(b) Suppose M is a finitely generated A-module. Prove that

$$d(M) = \dim_{A/\mathfrak{m}} M/\mathfrak{m}M,$$

where $M/\mathfrak{m}M$ is viewed as a vector space over $A/\mathfrak{m}$.

(Hint: It is clear that $d(M) \geq \dim_{A/\mathfrak{m}} M/\mathfrak{m}M$; now suppose $y_1 + \mathfrak{m}M, \ldots, y_d + \mathfrak{m}M$ is an $A/\mathfrak{m}$-basis of $M/\mathfrak{m}M$, and let N be the submodule of M that is generated by y_i's. Use part (a) for M/N.)

(c) (f.g. projective $\Rightarrow$ locally free) Suppose P is a finitely generated projective A-module. Prove that A is free.

(Hint: Suppose $d(P) = d$; so there is a S.E.S.

$$0 \to N \to A^d \to P \to 0.$$

Since P is projective, we have that there is an A-module isomorphism $\phi : A^d \xrightarrow{\sim} P \oplus N$. Show that $\phi(\mathfrak{m}A^d) = \mathfrak{m}P \oplus \mathfrak{m}N$; and then use part (b).)

(Remark. This exercise implies that for an arbitrary unital commutative ring A, a finitely generated projective module P is locally free; that means for any $\mathfrak{p} \in \mathrm{Spec}(A)$, $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module. The converse of this statement is true as well: a f.g. locally free module is projective.)

3. Suppose A is an integral domian.

   (a) Show that a submodule of a finitely generated free A-module is a free A-module if and only if A is a PID.

   (b) Suppose $(A, \mathfrak{m})$ is a Noetherian local ring, as well. Show that a submodule of a finitely generated projective A-module is projective if and only if A is a PID.

(c) Is a submodule of a finitely generated projective A-module necessarily projective?

(Hint: show that $k[x_1, x_2]_{\langle x_1, x_2 \rangle}$ is a local Noetherian integral domain which is not a PID; or come up with your own example.)